Transitivity of automorphism groups of Gizatullin surfaces

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ABSTRACT. We show that the automorphism group of a certain subclass of smooth Gizatullin surfaces with a distinguished and rigid extended divisor is generated by automorphisms of $\mathbb{A}^1$-fibrations. Moreover, such surfaces yield examples of smooth Gizatullin surfaces with a non-transitive action of the automorphism group. Thus, they represent counterexamples to Gizatullin's conjecture. For such surfaces we give an explicit orbit decomposition of the natural action of the automorphism group.

Contents

1. Introduction

Gizatullin surfaces were introduced by Danilov and Gizatullin ([DG1], [DG2]). We recall that the Makar-Limanov invariant $\mathrm{ML}(V)$ of a normal affine surface V is defined by

$$\mathrm{ML}(V) = \bigcap_{\partial \in \mathrm{LND}(\mathbb{C}[V])} \ker(\partial).$$

A useful characterization of normal affine surfaces with trivial Makar-Limanov invariant is the following result due to Gizatullin ([Gi] II, Theorems 2 and 3), Bertin ([Be], Theorem 1.8), Bandman and Makar-Limanov ([BML]) in the smooth case and due to Dubouloz ([Du]) in the normal case:

Proposition 1.1. *(cf. [FKZ2], Theorem 4.3) For a normal affine surface V that is non-isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ or to $\mathbb{C}^* \times \mathbb{A}^1$, the following conditions are equivalent:*

(1) *$ML(V)$ is trivial, i. e. $ML(V) = \mathbb{C}$.*
(2) *The automorphism group acts on V with an open orbit, such that the complement is finite (such orbits are called* big*).*
(3) *V admits a smooth completion by a zigzag D. In other words, $V = X \backslash D$, where X is a complete surface and D is a linear chain of smooth rational curves.*

Normal affine surfaces V satisfying one of the equivalent conditions of Proposition 1.1 are called *Gizatullin surfaces*. In particular, the automorphism group of Gizatullin surface V is quite large compared to surfaces in general. Studying simple smooth Gizatullin surfaces, like the affine plane $\mathbb{A}^2$ or Danielewski surfaces $V_P = \{xy - P(z) = 0\} \subseteq \mathbb{A}^3$ with a polynomial P having pairwise distinct roots, one easily sees that the automorphism group acts transitively on these surfaces. In this case, the big orbit O coincides with V. Cleary, in the singular case O cannot coincide with V. However, it is still an open question whether O coincides with V in

the smooth case. More generally Gizatullin formulated the following conjecture in [Gi] II:

Conjecture (Gizatullin): Let V be a smooth Gizatullin surface. Then the action of the automorphism group is always transitive on V. In other words, that is, V coincides with the big orbit O of $\mathrm{Aut}(V)$.

In general it is difficult to determine the orbits of the natural action of $\mathrm{Aut}(V)$ for general normal affine varieties V. But there are some nice results in higher dimension. For example, in [DM-JP] it is shown that the automorphism group of the Koras-Russell cubic $X = \{x + x^2y + z^2 + t^3 = 0\} \subseteq \mathbb{A}^4$ admits exactly 4 orbits, one of them being the fix point $p = (0,0,0,0)$.

The main aim of this article is to construct families of smooth Gizatullin surfaces V, satisfying special conditions and to determine the orbit decomposition of the natural action of $\mathrm{Aut}(V)$. In particular, it turns out that for these surfaces the big orbit O is a proper subset of V. It follows that such Gizatullin surfaces provide counterexamples to the Gizatullin conjecture.
Let (X, D) be a SNC-completion of a Gizatullin surface V so that $V = X \backslash D$ and D is a simple normal crossing divisor. It is well known that D can be transformed by birational transformations into *standard form*. This means that $D = C_0 \cup \cdots \cup C_n$ is a chain of smooth rational curves satisfying $C_0^2 = C_1^2 = 0$ and $C_i^2 \le -2$ for $i \ge 2$. The standard form of the boundary divisor D is (up to reversion) an invariant of the abstract isomorphism type of V. But in general this invariant is too weak. It is more convenient to introduce a stronger invariant, the so called *extended divisor* D_{ext}, defined as follows. The curves C_0 and C_1 provide $\mathbb{P}^1$-fibrations $\Phi_0 := \Phi_{|C_0|} : X \to \mathbb{P}^1$ and $\Phi_1 := \Phi_{|C_1|} : X \to \mathbb{P}^1$. These $\mathbb{P}^1$-fibrations lift to the minimal resolution of singularities $\tilde{X}$ of X. By [FKZ4], Lemma 2.19, Φ_0 admits at most one degenerate fiber, without lost of generality the fiber over 0, and we introduce the *extended divisor* to be

$$D_{\mathrm{ext}} := C_0 \cup C_1 \cup \Phi_0^{-1}(0).$$

The extended divisor D_{ext} always contains the boundary divisor D. The connected components of $D_{\mathrm{ext}} - D$ are called *feathers*. We denote them by $F_{i,j}$, $2 \le i \le 2$, $1 \le j \le r_i$ and assume that $F_{i,j}$ is attached to the curve C_i. In fact, in the smooth case the feathers are irreducible. The Matching Principle (cf. [FKZ6]) states that there is a natural bijection between feathers $F_{i,j}$ of (X, D) and feathers $F_{i,j}^\vee$ of the completion $(X^\vee, D^\vee)$ obtained by reversing the boundary zigzag. Our candidates for potential counterexamples for the Gizatullin conjecture are smooth Gizatullin surfaces that admit a so called *distinguished* and *rigid* extended divisor. Indeed, if one extended divisor of V has this property, then every extended divisor of V does. To be more precise, we show the following result:

Theorem 1.2. *Let V be a smooth Gizatullin surface, (X, D) a completion in standard form such that the extended divisor D_{ext} is distinguished and rigid. Let $A_i = \{P_{i,1}, \ldots, P_{i,r_i}\} \subseteq C_i \backslash (C_{i-1} \cup C_{i+1}) \cong \mathbb{C}^*$, $3 \le i \le n-1$, be the base point set of the feathers $F_{i,j}$. For a finite subset $A \subseteq \mathbb{C}^*$, we denote by $G(A)$ the group $\{\alpha \in \mathbb{C}^* \mid \alpha \cdot A = A\}$. Moreover, for $4 \le i \le n-2$ we let $B_{i,1}, \ldots, B_{i,m_i}$ be the orbits of the $G(A_i)$-action on A_i,*

$$O_{i,j} := \bigcup_{1 \le l \le r_i; P_{i,l} \in B_{i,j}} F_{i,l} \cap F_{i,l}^\vee \subseteq V, \quad 1 \le j \le m_i,$$

and

$$O_0 := V \backslash \left(\bigcup_{4 \le i \le n-2, 1 \le j \le r_i} F_{i,j} \cap F_{i,j}^\vee \right).$$

Then the following hold:

(1) *The set* O_0 *is the big orbit of the action of* $\operatorname{Aut}(V)$ *on* V *and the subsets* $O_{i,j}$ *are invariant under* $\operatorname{Aut}(V)$.
(2) *If at most two of the* r_i *are non-zero, then* O_0 *and the* $O_{i,j}$ *form the orbit decomposition of the natural action of the automorphism group* $\operatorname{Aut}(V)$ *on* V.

Furthermore, we show that in the generic case the automorphism group of such surfaces V is generated by automorphisms of $\mathbb{A}^1$-fibrations, that is, by automorphisms which preserve certain $\mathbb{A}^1$-fibrations. We also give an explicit description of $\operatorname{Aut}(V)$ as an amalgamated product of two automorphism subgroups.

This article is structured as follows. In section 2 we introduce the main tools employed to work with Gizatullin surfaces and more generally with $\mathbb{A}^1$-fibered surfaces. We are mainly interested in presentations and properties of standard and 1-standard completions of such surfaces. In particular, we give a decomposition of birational maps between 1-standard pairs. We also introduce the Matching Principle and the rigidity of extended divisors. Finally, we give a concrete description of smooth Gizatullin surfaces in local affine coordinates.
In section 3 we apply these methods to show that the automorphism group of a general smooth Gizatullin surface V with a distinguished and rigid extended divisor is generated by automorphisms of $\mathbb{A}^1$-fibrations. Moreover, for such surfaces we give the orbit decomposition of the action of the automorphism group of V. In particular we show that the automorphism group admits fix points in general. These surfaces provide counterexamples to the Gizatullin conjecture. In subsection 3.3 we give explicit presentations of the automorphism groups of such surfaces as amalgamated products of two automorphism subgroups.
Finally, in Section 4 we deal with the singular case and we show that similar results hold.

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2. Preliminaries

In the following we consider surfaces over the field $\mathbb{K} = \mathbb{C}$ of complex numbers. All results also hold for arbitrary algebraically closed fields of characteristic zero.

2.1. $\mathbb{A}^1$-fibered surfaces and Gizatullin surfaces.

Definition 2.1. *A* zigzag D *on a normal projective surface* X *is an SNC-divisor supported in the smooth locus* X_{reg} *of* X, *with irreducible components isomorphic to* $\mathbb{P}^1$ *and whose dual graph is a chain. If* $supp(D) = \bigcup_{i=0}^n C_i$ *is the decomposition into irreducible components, one can order the* C_i *such that*

$$C_i.C_j = \begin{cases} 1 & ,|i-j| = 1 \\ 0 & ,|i-j| > 1. \end{cases}$$

A zigzag with such an ordering is called oriented *and the sequence* $[[(C_0)^2, \ldots, (C_n)^2]]$ *is called the* type *of* D. *The same zigzag with the reverse ordering is denoted by* ${}^tD$.
An oriented sub-zigzag *of an oriented zigzag is an SNC-divisor* D' *with* $supp(D') \subseteq supp(D)$ *which is a zigzag for the induced ordering.*
We say that an oriented zigzag D *is composed of sub-zigzags* $Z_1, \ldots, Z_s$, *and following [BD] we denote* $D = Z_1 \triangleright \cdots \triangleright Z_s$, *if the* Z_i, $1 \le i \le s$, *are oriented sub-zigzags of* D *whose union is* D *and the components of* Z_i *precede those of* Z_j *for* $i < j$.

Surfaces completable by a zigzag were first studied by Danilov and Gizatullin (cf. [DG1] and [DG2]).

Definition 2.2. *Normal affine surfaces V satisfying one of the conditions in Proposition 1.1 are called* Gizatullin surfaces.

For the rest of this article we fix the following notation:

Notation: If V is a Gizatullin surface and (X, D) is a completion of V by a zigzag D, then

$$D = C_0 + \cdots + C_n \quad \text{and} \quad C_i \text{ and } C_j \text{ have a non-empty intersection only for } |i - j| = 1.$$

Given a Gizatullin surface V together with a completion (X, D) by a zigzag, we can always associate a linear weighted graph Γ_D to (X, D). The vertices v_i, $0 \le i \le n$, are boundary components C_i and the weights are the corresponding self-intersections $w_i := C_i^2$. Thus Γ_D has the form

$$\Gamma_D : \begin{array}{ccccccc} C_0 & & C_1 & & & & C_n \\ \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ \\ w_0 & & w_1 & & & & w_n \end{array} .$$

For a better systematic understanding of Gizatullin surfaces we introduce elementary transformations of weighted graphs.

Definition 2.3. *Given an at most linear vertex v of a weighted graph Γ with weight 0 one can perform the following transformations. If v is linear with neighbors v_1, v_2 then we blow up the edge connecting v and v_1 in Γ and blow down the proper transform of v:*

$$(2.1) \qquad \begin{array}{ccccc} & v_1 & v' & v_2 & \\ \cdots & \circ\text{---} & \circ\text{---} & \circ & \cdots \\ & w_1 - 1 & 0 & w_2 + 1 & \end{array} \dashrightarrow \begin{array}{cccccc} & v_1 & v' & v & v_2 & \\ \cdots & \circ\text{---} & \circ\text{---} & \circ\text{---} & \circ & \cdots \\ & w_1 - 1 & -1 & -1 & w_2 & \end{array} \rightarrow \begin{array}{cccc} & v_1 & v & v_2 \\ \cdots & \circ\text{---} & \circ\text{---} & \circ \\ & w_1 & 0 & w_2 \end{array}$$

Similarly, if v is an end vertex of Γ connected to the vertex v_1 then one proceeds as follows:

$$(2.2) \qquad \begin{array}{ccc} & v_1 & v' \\ \cdots & \circ\text{---} & \circ \\ & w_1 - 1 & 0 \end{array} \dashrightarrow \begin{array}{cccc} & v_1 & v' & v \\ \cdots & \circ\text{---} & \circ\text{---} & \circ \\ & w_1 - 1 & -1 & -1 \end{array} \rightarrow \begin{array}{ccc} & v_1 & v \\ \cdots & \circ\text{---} & \circ \\ & w_1 & 0 \end{array}$$

These operations (2.1) and (2.2) and their inverses are called elementary transformations *of* Γ. *If such an elementary transformation involves only an inner blowup then we call it* inner. *Thus (2.1) and (2.2) are inner whereas the inverse of (2.2) is not as it involves an outer blowup.*

We consider a Gizatullin surface $V = X \backslash D$, where X is projective and D is a zigzag. By a sequence of blowups and blowdowns we can transform the dual graph Γ_D of D into *standard form*, i. e. we can achieve that $C_0^2 = C_1^2 = 0$ and $C_i^2 \le -2$ for all $i \ge 2$ (cf. [DG1], [FKZ3], [Da]). Moreover, this representation is unique up to reversion meaning that for two standard forms $[[0, 0, w_2, \ldots, w_n]]$ and $[[0, 0, w'_2, \ldots, w'_n]]$ either $w_i = w'_i$ or $w_i = w'_{n+2-i}$ holds (cf. [FKZ3]).
The reversion process can be described easily. We start with a boundary divisor of type $[[0, 0, w_2, \ldots, w_n]]$. Performing the elementary transformation (2.1) at the vertex corresponding to C_1 we obtain a boundary divisor of type $[[-1, 0, w_2 + 1, w_3, \ldots, w_n]]$. Repeating this procedure finitely many times we obtain a boundary divisor of type $[[w_2, 0, 0, w_3, \ldots, w_n]]$. This means that we can move pairs of zeros to the right. Repeating this, we finally obtain a boundary divisor of type $[[w_2, \ldots, w_n, 0, 0]]$. Notice that all birational transformations are centered in the boundary, i. e. these transformations yield isomorphisms on the affine parts.

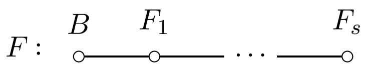
Definition 2.4. *A zigzag D on a normal projective surface X is called* m-standard *(or in* m-standard form*), if it is of type $[[0, -m, w_2, \ldots, w_n]]$ with $n \geq 1$ and $w_i \leq -2$ (in the case of $n = 1$ there are no weights w_i).*
An m-standard pair *is a pair (X, D) consisting of a normal projective surface X and an m-standard zigzag D on X. If $m = 0$, then (X, D) is called a* standard pair. *A* birational map *$\varphi : (X, D) \dashrightarrow (X', D')$ between m-standard pairs is a birational map $\varphi : X \dashrightarrow X'$ which restricts to an isomorphism $\varphi|_{X \backslash D} : X \backslash D \xrightarrow{\sim} X' \backslash D'$.*

Since the underlying projective surface X of an m-standard pair is rational, it is equipped with a rational fibration $\overline{\pi} = \Phi_{|C_0|} : X \to \mathbb{P}^1$ defined by the complete linear system $|C_0|$. In particular, if $m = 0$, there are even two $\mathbb{P}^1$-fibrations $\Phi_0 := \Phi_{|C_0|}, \Phi_1 := \Phi_{|C_1|} : X \to \mathbb{P}^1$ and, thus, a morphism

$$\Phi := \Phi_0 \times \Phi_1 : X \to \mathbb{P}^1 \times \mathbb{P}^1,$$

which is birational (cf. [FKZ4], Lemma 2.19). After a change of coordinates we can assume that $C_0 = \Phi_0^{-1}(\infty)$, $\Phi(C_1) = \mathbb{P}^1 \times \{\infty\}$ and $C_2 \cup \cdots \cup C_n \subseteq \Phi_0^{-1}(0)$. The divisor $D_{\text{ext}} := C_0 \cup C_1 \cup \Phi_0^{-1}(0)$ is called the *extended divisor*. Before determining the structure of the extended divisor, we introduce the notion of a *feather*:

Definition 2.5. (1) *A* feather *is a linear chain*



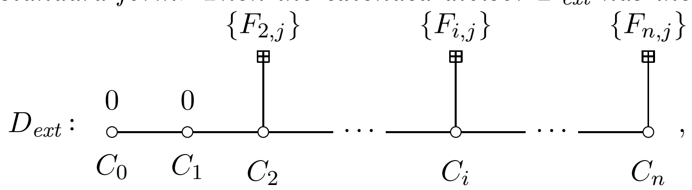
of smooth rational curves such that $B^2 \leq -1$ and $F_i^2 \leq -2$ for all $i \geq 1$. B is called the bridge curve.

(2) *A collection of feathers $\{F_\rho\}$ consists of feathers F_ρ, $1 \leq \rho \leq r$, which are pairwise disjoint. Such a collection will be denoted by a plus box*

$$\overset{\{F_\rho\}}{\boxplus}\ .$$

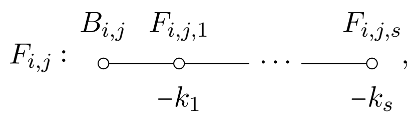
(3) *Let $D = C_0 + \cdots + C_n$ be a zigzag. A collection $\{F_\rho\}$ is* attached to a curve C_i *if the bridge curves B_ρ meet C_i in pairwise distinct points and all the feathers F_ρ are disjoint with the curves C_j for $j \neq i$.*

Lemma 2.6. *(cf. [FKZ5], Prop. 1.11) Let $(\tilde{X}, D)$ be a minimal SNC completion of the minimal resolution of singularities of a Gizatullin surface V. Furthermore, let $D = C_0 + \cdots + C_n$ be the boundary divisor in standard form. Then the extended divisor D_{ext} has the dual graph*



where $\{F_{i,j}\}$, $1 \leq j \leq r_i$, are feathers attached to the curve C_i. Moreover, $\tilde{X}$ is obtained from $\mathbb{P}^1 \times \mathbb{P}^1$ by a sequence of blowups with centers in the images of the components C_i, $i \geq 2$.

Remark 2.7. We consider the feathers $F_{i,j} := B_{i,j} + F_{i,j,1} + \cdots + F_{i,j,k_{i,j}}$ mentioned in Lemma 2.6. The collection of linear chains $R_{i,j} := F_{i,j,1} + \cdots + F_{i,j,k_{i,j}}$ corresponds to the minimal resolution of singularities of V. Thus, if (X, D) is a standard completion of V and $(\tilde{X}, D)$ is the minimal resolution of singularities of (X, D), the chain $R_{i,j}$ contracts via $\mu : (\tilde{X}, D) \to (X, D)$ to a singular point of V. Moreover, one can classify the type of these singularities. For a feather



$k_i \geq 2$,

we define

$$\frac{d}{e} := [k_1, \ldots k_s] := k_1 - \cfrac{1}{k_2 - \cfrac{1}{\cdots - \frac{1}{k_s}}}.$$

The chain $R_{i,j}$ contracts via μ to a *cyclic quotient singularity of type* (d, e), that is, to a singularity analytically isomorphic to the singular point of the toric surface $V_{d,e} := \mathbb{A}^2/\mathbb{Z}_d$, where the action of $\mathbb{Z}_d = \langle\zeta\rangle$ ($\zeta \in \mathbb{C}^*$ a primitive d-th root of unity) on $\mathbb{A}^2$ is given by $\zeta.(x, y) = (\zeta x, \zeta^e y)$ (cf. [Hi]). In partcular, V has at most cyclic quotient singularities (cf. [Mi], §3, Lemma 1.4.4 (1)).
In particular, V is smooth if and only if every $R_{i,j}$ is empty, i. e. if every feather $F_{i,j}$ is irreducible and reduces to a single bridge curve $B_{i,j}$ (cf. [FKZ5], 1.8, 1.9 and Remark 1.12).

In connection with Lemma 2.6 we abbreviate the subdivisor $\sum_{k\geq i} C_k + \sum_{j_k;k\geq i} F_{k,j_k}$ by $D_{\text{ext}}^{\geq i}$ and the subdivisor $\sum_{k>i} C_k + \sum_{j_k;k\geq i} F_{k,j_k} = D_{\text{ext}}^{\geq i} \ominus C_i$ by $D_{\text{ext}}^{>i}$.

Similarly as standard completions of Gizatullin surfaces arise from $\mathbb{P}^1 \times \mathbb{P}^1$, 1-standard completions arise from the Hirzebruch surface $\mathbb{F}_1$. More explicit, we have the following lemma:

Lemma 2.8. *(cf. [BD], Lemma 1.0.7) Let (X, D) be a 1-standard pair and let $\mu : Y \to X$ be the minimal resolution of singularities of X. Then there exists a birational morphism $\eta : Y \to \mathbb{F}_1$, unique up to an automorphism of $\mathbb{F}_1$, that restricts to an isomorphism outside the degenerate fibers of $\bar{\pi} \circ \mu$, and a commutative diagram*

$$\begin{tikzcd}
 & Y \arrow[dl, "\mu"'] \arrow[dr, "\eta"] \arrow[dd, "\mu\circ\bar{\pi}"] & \\
X \arrow[dr, "\bar{\pi}"'] & & \mathbb{F}_1 \arrow[dl, "\rho"] \\
 & \mathbb{P}^1 &
\end{tikzcd}.$$

Moreover, if $(X', D', \bar{\pi}')$ is another 1-standard pair with associated morphism $\eta' : Y' \to \mathbb{F}_1$, then $(X, D, \bar{\pi})$ and $(X', D', \bar{\pi}')$ are isomorphic if and only if there exists an automorphism of $\mathbb{F}_1$ isomorphically mapping $\eta(\mu_*^{-1}(C_0))$ onto $\eta'(\mu_*'^{-1}(C_0'))$ and isomorphically sending the base-points of η^{-1} (including infinitely near ones) onto those of η'^{-1}.

The Hirzebruch surface $\mathbb{F}_1$ is just the blowup of $\mathbb{P}^2$ in a single point, say $(1 : 0 : 0)$. Thus, $\mathbb{F}_1$ is given by

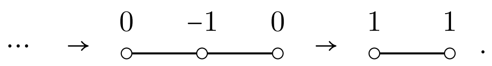
$$\mathbb{F}_1 = \{((x : y : z), (s : t)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yt - zs = 0\}.$$

The $\mathbb{P}^1$-fibration mentioned in Lemma 2.8 is just the projection onto the second factor, i. e.

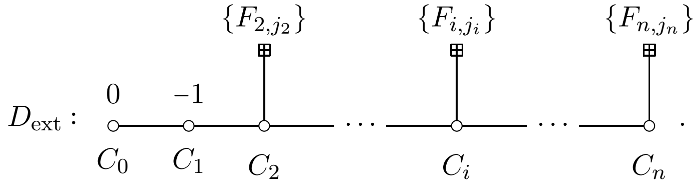
$$\rho : \mathbb{F}_1 \to \mathbb{P}^1, \quad ((x : y : z), (s : t)) \mapsto (s : t).$$

Moreover, on $\mathbb{F}_1 \backslash (C_0 \cup C_1) \cong \mathbb{A}^2$ we can introduce the affine coordinates $x_0 := x/z$ and $y_0 := y/z$. These coordinates become important in the next section.

Since $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ in one point, every 1-standard pair (X, D) arises as a blowup of $\mathbb{P}^2$ and the blowup process starts as follows



Here we can take any two lines in $\mathbb{P}^2$ for the two curves with self-intersection 1. The extended divisor can also be defined for 1-standard pairs and D_{ext} becomes



This results in the same divisor as taking the extended divisor of the corresponding standard completion, blowing up the intersection point $C_0 \cap C_1$ and blowing down the proper transform of C_0.
We will often deal with 1-standard pairs. It follows from [BD], Lemma 2.1.1 that every birational map $\varphi : (X, D) \dashrightarrow (X', D')$ between 1-standard pairs which is not an isomorphism has a unique base point $p \in C_0$. This base point is called the *center* of φ. In general, this yields qualitatively different maps depending on whether $p \in C_0 \cap C_1$ or $p \in C_0 \backslash C_1$.

Definition 2.9. *Two $\mathbb{A}^1$-fibered surfaces (V, π) and (V', π') are said to be isomorphic if there exists an isomorphism $\Psi : V \to V'$ and an automorphism ψ of $\mathbb{A}^1$, such that $\pi' \circ \Psi = \psi \circ \pi$.*
Two $\mathbb{A}^1$-fibrations π, π' on a surface V are said to be isomorphic if (V, π) and (V, π') are isomorphic.

As mentioned above, there are two basic types of birational maps between 1-standard pairs: The *fibered modifications*, which preserve the given $\mathbb{A}^1$-fibrations, and the *reversions*, which are, in some sense, the simplest maps that do not preserve the given fibrations.

Definition 2.10. *Let $\varphi : (X, D) \dashrightarrow (X', D')$ be a birational map between 1-standard pairs and let $D = C_0 \rhd \cdots \rhd C_n$ and $D' = C_0' \rhd \cdots \rhd C_n'$ be the oriented boundary divisors.*
(1) *(Fibered modification) φ is called a* fibered map *if it restricts to an isomorphism of $\mathbb{A}^1$-fibered quasi-projective surfaces*

$$\begin{tikzcd} V = X\backslash D \arrow[r, "\sim", "\varphi"'] \arrow[d, "\bar{\pi}|_V"'] & V' = X'\backslash D' \arrow[d, "\bar{\pi}'|_{V'}"] \\ \mathbb{A}^1 \arrow[r, "\sim"] & \mathbb{A}^1. \end{tikzcd}$$

φ is called fibered modification *if it is not an isomorphism.*
(2) *(Reversion) φ is called* reversion *if it admits a resolution of the form*

$$\begin{tikzcd}[column sep=small] & (Z, \tilde{D} = C_n \rhd \cdots \rhd C_1 \rhd H \rhd C_1' \rhd \cdots \rhd C_{n'}') \arrow[dl, "\sigma"'] \arrow[dr, "\sigma'"] & \\ (X, {}^tD) \arrow[rr, dashed, "\varphi"] & & (X', D'), \end{tikzcd}$$

where H is a zigzag with boundaries C_0 (left) and C_0' (right) and where $\sigma : Z \to X$ and $\sigma' : Z \to X'$ are smooth contractions of the sub-zigzags $H \rhd C_1' \rhd \cdots \rhd C_{n'}'$ and $C_n \rhd \cdots \rhd C_1 \rhd H$ of $\tilde{D}$ onto C_0 and C_0' respectively.

Remark 2.11. We already introduced the notion of a reversion for standard pairs. Moreover, given a standard completion (X, D) of V, we have a birational map $(X, D) \dashrightarrow (X', D')$ to a 1-standard completion (X', D') of V (see above). We will see below (cf. Proposition 2.13) that the two notions of a reversion coincide after performing such elementary transformations on the boundary.

It turns out that fibered modifications are just the liftings of appropriate triangular automorphisms of $\mathbb{A}^2$:

Lemma 2.12. *(cf. [BD], Lemma 2.2.3) Let $\varphi : (X, D, \bar{\pi}) \dashrightarrow (X', D', \bar{\pi}')$ be a birational map between 1-standard pairs and let $X \overset{\mu}{\leftarrow} Y \overset{\eta}{\rightarrow} \mathbb{F}_1$ and $X' \overset{\mu'}{\leftarrow} Y' \overset{\eta'}{\rightarrow} \mathbb{F}_1$ be as in Lemma 2.8. Then the following are equivalent:*

(1) *φ restricts to an isomorphism $(X\backslash D,\pi)\xrightarrow{\sim}(X'\backslash D',\pi')$.*
(2) *$(\mu')^{-1}\circ\varphi\circ\mu : Y \dashrightarrow Y'$ is the lift via η and η' of an isomorphism of affine $\mathbb{A}^1$-fibered surfaces*

$$\begin{tikzcd} \mathbb{A}^2=\mathbb{F}_1\backslash(\eta(C_0)\cup\eta(C_1)) \arrow[r, "\sim", "\Psi"'] \arrow[d, "\rho|_{\mathbb{A}^2}"'] & \mathbb{A}^2=\mathbb{F}_1\backslash(\eta'(C_0)\cup\eta'(C_1)) \arrow[d, "\rho|_{\mathbb{A}^2}"] \\ \mathbb{A}^1 \arrow[r, "\sim", "\psi"'] & \mathbb{A}^1, \end{tikzcd}$$

which maps isomorphically the base points of η^{-1} onto those of $(\eta')^{-1}$. The map Ψ is of the form $\Psi(x_0,y_0)=(ax_0+P(y),by+c)$ with $a,b\in\mathbb{C}^*$, $c\in\mathbb{C}$ and $P(y)\in\mathbb{C}[y]$.

Moreover, $\varphi:(X,D)\dashrightarrow(X',D')$ is an isomorphism if and only if Ψ is affine.

Indeed, the center p gives the full control over the reversion:

Proposition 2.13. *(Uniqueness of reversions, cf. [BD], Prop. 2.3.7) For every 1-standard pair (X,D) and every point $p\in C_0\backslash C_1$ there exist a 1-standard pair (X',D') and a reversion $\varphi:(X,D)\dashrightarrow(X',D')$, unique up to an isomorphism at the target, having p as a unique proper base point. Moreover, if $\Gamma_D=[[0,-1,w_2,\dots,w_n]]$, then $\Gamma_{D'}=[[0,-1,w_n,\dots,w_2]]$.*

These two types of maps, the fibered modifications and the reversions, differ in the position of their center:

Lemma 2.14. *(cf. [BD], Lemma 2.4.1) Let $\varphi:(X,D)\dashrightarrow(X',D')$ be a birational map between 1-standard pairs.*

(a) *If φ is a fibered modification, it is centered at $p=C_0\cap C_1$, and C_0' is the only irreducible component of D' contracted by φ^{-1}.*
(b) *If φ is a reversion, it is centered at $p=C_0\backslash C_1$, and φ^{-1} contracts the curves C_i', $i\geq 1$, on p as well as C_0' on p if and only if $C_i'^2\leq -3$ holds for some $i\geq 2$.*

If the type of the sub-zigzag $C_2\rhd\cdots\rhd C_n$ is not a palindrome, then the composition of two reversions cannot be a reversion. Otherwise, we have the following

Lemma 2.15. *(cf. [BD], Lemma 2.3.8) For $i=1,2$ let $\varphi_i:(X,D)\dashrightarrow(X_i,D_i)$ be a reversion of 1-standard pairs and assume that every component of D has self-intersection ≥ -2. If the proper base points of φ_1 and φ_2 are distinct (equal, respectively), then the map $\varphi_2\circ\varphi_1^{-1}$ is a reversion (an isomorphism, respectively).*

The key observation to control birational maps between 1-standard pairs is to decompose any such map into fibered modifications and reversions:

Theorem 2.16. *(cf. [BD], Theorem 3.0.2) Let $\varphi:(X,D)\dashrightarrow(X',D')$ be a birational map between 1-standard pairs restricting to an isomorphism $X\backslash D\xrightarrow{\sim}X'\backslash D'$. If φ is not an isomorphism, then it can be decomposed into a finite sequence*

$$\varphi=\varphi_n\circ\cdots\circ\varphi_1:(X,D)=(X_0,D_0)\overset{\varphi_1}{\to}(X_1,D_1)\overset{\varphi_2}{\to}\cdots\overset{\varphi_n}{\to}(X_n,D_n)=(X',D')$$

of fibered modifications and reversions between 1-standard pairs (X_i,D_i). Moreover, such a factorization of minimal length is unique, meaning, if

$$\varphi=\varphi_n'\circ\cdots\circ\varphi_1':(X,D)=(X_0',D_0')\overset{\varphi_1'}{\to}(X_1',D_1')\overset{\varphi_2'}{\to}\cdots\overset{\varphi_n'}{\to}(X_n',D_n')=(X',D')$$

is another factorization of minimal length, then there exist isomorphisms of 1-standard pairs $\alpha_i:(X_i,D_i)\to(X_i',D_i')$, such that $\alpha_i\circ\varphi_i=\varphi_i'\circ\alpha_{i-1}$ for $i=2,\dots,n$.

2.2. **Automorphisms of $\mathbb{A}^1$-fibrations and associated graphs.** For an $\mathbb{A}^1$-fibered surface V we introduce a (not necessarily finite) graph $\mathcal{F}_V$ which reflects the structure of the automorphism group of V.

Definition 2.17. *To every normal quasi-projective surface V we associate the oriented graph $\mathcal{F}_V$ as follows:*

(1) *A vertex of $\mathcal{F}_V$ is an equivalence class of a 1-standard pair (X, D), such that $X\backslash D \cong V$, where two 1-standard pairs $(X_1, D_1, \bar{\pi}_1)$ and $(X_2, D_2, \bar{\pi}_2)$ define the same vertex if and only if $(X_1\backslash D_1, \pi_1) \cong (X_2\backslash D_2, \pi_2)$.*
(2) *An arrow of $\mathcal{F}_V$ is an equivalence class of reversions. If $\varphi : (X, D) \to (X', D')$ is a reversion, then the class $[\varphi]$ of φ is an arrow starting from $[(X, D)]$ and ending at $[(X', D')]$. Two reversions $\varphi_1 : (X_1, D_1) \dashrightarrow (X_1', D_1')$ and $\varphi_2 : (X_2, D_2) \dashrightarrow (X_2', D_2')$ define the same arrow if and only if there exist isomorphisms $\theta : (X_1, D_1) \to (X_2, D_2)$ and $\theta' : (X_1', D_1') \to (X_2', D_2')$, such that $\varphi_2 \circ \theta = \theta' \circ \varphi_1$.*

Remark 2.18. It is easy to see that for a 1-standard pair (X, D) two reversions $\varphi : (X, D) \dashrightarrow (X_1, D_1)$ and $\varphi : (X, D) \dashrightarrow (X_2, D_2)$ with centers in p_1 and p_2 respectively define the same arrow if and only if there exists an automorphism $\psi \in \mathrm{Aut}(X, D)$ such that $\psi(p_1) = p_2$. Thus, for a *fixed* completion (X, D) of V we have a bijection between the arrows of $\mathcal{F}_V$, starting at (X, D) and orbits of the action of $\mathrm{Aut}(X, D)$ on $C_0\backslash C_1$.

The structure of the graph $\mathcal{F}_V$ allows us to decide, whether the automorphism group $\mathrm{Aut}(V)$ of V is generated by automorphisms of $\mathbb{A}^1$-fibrations. Here we say that $\varphi \in \mathrm{Aut}(V)$ is an *automorphism of $\mathbb{A}^1$-fibrations* if there exists an $\mathbb{A}^1$-fibration $\pi : V \to \mathbb{A}^1$, such that φ induces an isomorphism $\varphi : (V, \pi) \xrightarrow{\sim} (V, \pi)$.

Proposition 2.19. *(cf. [BD], Prop. 4.0.7) Let V be a normal quasi-projective surface with a non-empty graph $\mathcal{F}_V$. Then the following holds:*

(1) *The graph $\mathcal{F}_V$ is connected.*
(2) *There is a natural bijection between the set of vertices of $\mathcal{F}_V$ and the isomorphism classes of $\mathbb{A}^1$-fibrations on V.*
(3) *Let (X, D) be a 1-standard pair with $X\backslash D \cong V$ and let D contain at least one curve with self-intersection ≤ -3. Then the graph $\mathcal{F}_V$ is a tree if and only if $\mathrm{Aut}(V)$ is generated by automorphisms of $\mathbb{A}^1$-fibrations on V. Moreover, there is a natural exact sequence*
$$1 \to H \to \mathrm{Aut}(V) \to \Pi_1(\mathcal{F}_V) \to 1,$$
where H is the (normal) subgroup of $\mathrm{Aut}(V)$ generated by all automorphisms of $\mathbb{A}^1$-fibrations and $\Pi_1(\mathcal{F}_V)$ is the fundamental group of the graph $\mathcal{F}_V$.

One can obtain better descriptions of the group $\mathrm{Aut}(V)$ by introducing the notion of a graph of groups.

Definition 2.20. *A* graph of groups *is a pair $(\mathcal{F}, \mathcal{G})$ such that $\mathcal{F}$ is a graph and $\mathcal{G}$ consists of a family of vertex groups $\{G_v \mid v \in V(\mathcal{F})\}$ and a family of edge groups $\{G_\sigma \mid \sigma \in E(\mathcal{F})\}$ satisfying the following conditions:*

(1) *For every edge it holds $G_\sigma = G_{\sigma^{-1}}$.*
(2) *For every edge σ there are monomorphisms $\kappa_\sigma : G_\sigma \to G_{s(\sigma)}$ and $\lambda_\sigma : G_\sigma \to G_{t(\sigma)}$ such that $\lambda_\sigma = \kappa_{\sigma^{-1}}$.*

A path *in $(\mathcal{F}, \mathcal{G})$ is a sequence $(g_0, \sigma_1, g_1, \ldots, \sigma_r, g_r)$, where $g_i \in G_{v_i}$ and $v_0, \sigma_1, v_1, \ldots, \sigma_r, v_r$ is a path in $\mathcal{F}$. The homotopy equivalence relation $\simeq$ is the equivalence relation generated by the elementary homotopy equivalence relations $(\sigma, \lambda_\sigma(h), \sigma^{-1}, (\kappa_\sigma(h))^{-1}) \simeq (1)$ with $1 \in G_{s(\sigma)}$ and $(g, \sigma, 1, \sigma^{-1}, g') \simeq (gg')$. If v is a vertex of $\mathcal{F}$ then the homotopy classes of closed paths starting and ending in v form a group under the concatenation $(\ldots, g)(g', \ldots) = (\ldots, gg', \ldots)$. We denote this group by $\pi_1(\mathcal{F}, \mathcal{G}, v)$ and call it the* fundamental group *of $(\mathcal{F}, \mathcal{G})$ in v.*

We can equip $\mathcal{F}_V$ in a natural way with a structure of a graph of groups.

Definition 2.21. *Let V be a normal quasi-projective surface and let $\mathcal{F}_V$ be its associated graph. Then $\mathcal{F}_V$ admits a structure of a graph of groups by the following choice:*

(1) *For any vertex v of $\mathcal{F}_V$, fix a 1-standard pair $(X_v, D_v, \bar{\pi}_v)$ in the class v. The group G_v is equal to $\mathrm{Aut}(X_v \backslash D_v, \pi_v)$.*
(2) *For any arrow σ of $\mathcal{F}_V$, fix a reversion $r_\sigma : (X_\sigma, D_\sigma, \bar{\pi}_\sigma) \dashrightarrow (X'_\sigma, D'_\sigma, \bar{\pi'}_\sigma)$ in the class of σ and also an isomorphism $\mu_\sigma : (X'_\sigma \backslash D'_\sigma, \pi'_\sigma) \to (X_{t(\sigma)} \backslash D_{t(\sigma)}, \pi_{t(\sigma)})$ (by the index t we denote the target variety). Then the group G_σ is equal to*

$$\{(\varphi, \varphi') \in \mathrm{Aut}(X_\sigma, D_\sigma) \times \mathrm{Aut}(X'_\sigma, D'_\sigma) \mid r_\sigma \circ \varphi = \varphi' \circ r_\sigma\}$$

and the monomorphisms $\kappa_\sigma : G_\sigma \to G_{s(\sigma)}$ and $\lambda_\sigma : G_\sigma \to G_{t(\sigma)}$ are given by $\kappa_\sigma((\varphi, \varphi')) = \mu_{\sigma^{-1}} \circ \varphi \circ \mu_{\sigma^{-1}}^{-1}$ and $\lambda_\sigma((\varphi, \varphi')) = \mu_\sigma \circ \varphi' \circ \mu_\sigma^{-1}$ (index s denotes the source variety).

The first version of the following theorem was shown by Danilov and Gizatullin (cf. [DG], Theorem 5) and connects the structure of the graph of groups on $\mathcal{F}_V$ with the automorphism group of V:

Theorem 2.22. *(cf. [BD], Theorem 4.0.11) Let (X, D) be a 1-standard pair such that D admits at least one component with self-intersection ≤ -3 and let $V := X \backslash D$. If $\mathcal{F}_V$ is equipped with a structure of a graph of groups then the fundamental group of $\mathcal{F}_V$ is isomorphic to $\mathrm{Aut}(V)$.*

Remark 2.23. Let $\mathcal{F}$ be the graph $v \bullet \overset{\sigma}{\longleftrightarrow} \bullet\ w$ and $\mathcal{G} = \{(G_v, G_w), (G_\sigma)\}$. We can identify G_σ via λ_σ and κ_σ respectively with subgroups of G_v and G_w respectively. It is a well-known result that $\pi_1(\mathcal{F}, \mathcal{G}, v)$ is isomorphic to the amalgamated product $G_v \star_{G_\sigma} G_w$.

2.3. **The Matching Principle.** We consider a standard completion (X, D) of a smooth Gizatullin surface V as well as the reversed completion $(X^\vee, D^\vee)$ with $D = C_0 \cup \cdots \cup C_n$ and $D^\vee = C_0^\vee \cup \cdots \cup C_n^\vee$. We let $\Gamma_D = [[0, 0, w_2, \ldots, w_n]]$ and we denote the corresponding extended divisors by D_{ext} and $D^\vee_{\text{ext}}$, respectively. By inner elementary transformations we can move the pair of zeros to the right for several places. In this way we obtain, for every t, $2 \leq t \leq n+1$, a new completion (W, E) of V with boundary divisor $[[w_2, \ldots, w_{t-1}, 0, 0, w_t, \ldots, w_n]]$, i. e.

$$E = C_n^\vee \cup \cdots \cup C_{t^\vee}^\vee \cup C_{t-1} \cup C_t \cup \cdots \cup C_n,$$

if we identify $C_i \subseteq X$ and $C_j^\vee \subseteq X^\vee$ with their proper transforms in W. In particular, we can write E as $E = D^{\geq t-1} \cup D^{\vee \geq t^\vee}$ with new weights $C_{t-1}^2 = C_{t^\vee}^{\vee 2} = 0$. Moreover, there are natural isomorphisms

$$\begin{aligned} W \backslash D^{\vee \geq t^\vee} &= W \backslash (C_n^\vee \cup \cdots \cup C_{t^\vee}^\vee) \cong X \backslash (C_0 \cup \cdots \cup C_{t-2}), \\ W \backslash D^{\geq t-1} &= W \backslash (C_{t-1} \cup \cdots \cup C_n) \cong X^\vee \backslash (C_0^\vee \cup \cdots \cup C_{t^\vee - 1}^\vee). \end{aligned}$$

Definition 2.24. *The map*

$$\psi := \Phi_{|C_{t-1}|} : W \to \mathbb{P}^1$$

is called the correspondence fibration *for the pair $(C_t, C_{t^\vee}^\vee)$.*

There is a natural correspondence between feathers of D_{ext} and those of $D^\vee_{\text{ext}}$. The key observation is

Proposition 2.25. *(cf. [FKZ6], Lemma 3.3.4, Cor. 3.3.5, Lemma 3.3.6) Let $F_{i\rho}$ be a feather of D_{ext} attached to the component C_i. Then there exists a unique feather $F_{j\sigma}^\vee$ of $D^\vee_{ext}$ which intersects $F_{i\rho}$ in V and which is attached to a component $C_j^\vee$, such that $i + j \geq n + 2$. Moreover, $F_{i\rho}$ and $F_{j\sigma}^\vee$ intersect transversally and in a single point. If C_τ is the mother component of $F_{i\rho}$, then $C_{\tau^\vee}^\vee$ is the mother component of $F_{j\sigma}^\vee$.*

Definition 2.26. *Feathers $F_{i\rho}$ and $F^\vee_{j\sigma}$, which satisfy the conditions of Proposition 2.25, are called* matching feathers*.*

The condition $i+j \geq n+2$ is essential. Indeed, every feather $F_{t-1,\rho}$ is a section of ψ and therefore it meets every fiber of ψ. Since it cannot intersect $D^{\geq t}_{\text{ext}}$, it meets every feather $F^\vee_{t^\vee,\sigma}$ with $(F^\vee_{t^\vee,\sigma})^2 = -1$ on V.

Configuration spaces and the configuration invariant. Let V be a smooth Gizatullin surface with a standard completion (X, D). By a theorem of Gizatullin (cf. [Gi] or [FKZ3], Cor. 3.33), the sequence of weights $[[w_2, \dots, w_n]]$ (up to reversion) of the boundary divisor D is a discrete invariant of the abstract isomorphism type of V. In the following we introduce a stronger continuous invariant of V, the *configuration invariant.*

For a natural number $s \geq 1$ we denote the configuration space of all s-points subsets $\{\lambda_1, \dots, \lambda_s\} \subseteq \mathbb{A}^1$ by $\mathcal{M}^+_s$. We can identify $\mathcal{M}^+_s$ in a natural way with the Zariski open subset of $\mathbb{A}^s$:

$$\mathcal{M}^+_s \cong \mathbb{A}^s \backslash \{\operatorname{discr}(P) = 0\}, \quad \text{where} \quad P = \prod_{j=1}^{s} (X - \lambda_j).$$

The group $\operatorname{Aut}(\mathbb{A}^1)$ acts on $\mathcal{M}^+_s$ in a natural way. We let

$$\mathfrak{M}^+_s := \mathcal{M}^+_s / \operatorname{Aut}(\mathbb{A}^1).$$

Thus, $\mathfrak{M}^+_s$ is an $(s-2)$-dimensional affine variety.

Now, let $\mathcal{M}^*_s$ be the configuration space of all s-points subsets $\{\lambda_1, \dots, \lambda_s\} \subseteq \mathbb{C}^* = \mathbb{A}^1 \backslash \{0\}$. Similarly, the group $\mathbb{C}^*$ acts on $\mathcal{M}^*_s$ and we let

$$\mathfrak{M}^*_s := \mathcal{M}^*_s / \mathbb{C}^*.$$

Before introducing the configuration invariant we have to distinguish two types of boundary components.

Definition 2.27. (1) *For a natural number $i \in \{2, \dots, n\}$ s_i shall denote the number of feathers of D_{ext} whose mother component is C_i.*

(2) *The component C_i is called a* $*$-component *if*
  (i) *$D^{\geq i+1}_{ext}$ is not contractible and*
  (ii) *$D^{\geq i+1}_{ext} - F_{j,k}$ is not contractible for every feather $F_{j,k}$ of $D^{\geq i+1}_{ext}$ with mother component C_τ, where $\tau < i$.*

Otherwise C_i is called a $+$-component.

For example, C_2 and C_n are always $+$-components. In the following we let $\tau_i = *$ in the first case and $\tau_i = +$ in the second one.

It is easily seen that in the blowup process $\tilde{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ ($\tilde{X}$ is a standard completion of the minimal resolution of singularities V' of V) every $*$-component C_i, $3 \leq i \leq n-1$, appears as a result of an inner blowup of the previous zigzag, while an outer blowup of a zigzag creates a $+$-component.

The following lemma states that reversions do not change the type of a component:

Lemma 2.28. *(cf. [FKZ6], Lemma 3.3.10) C_t is a $*$-component if and only if $C^\vee_{t^\vee}$ is a $*$-component.*

Now we are able to construct the so-called *configuration invariant of V*. In the following, for a $+$-component C_i we construct a family of points p_{ij}, $1 \leq j \leq s_i$, on $C_i \backslash C_{i-1} \cong \mathbb{A}^1$. For every feather F_{ij} with self-intersection -1 we let p_{ij} be its intersection point with C_i. Moreover, if there exists a feather F_{kj} with mother component C_i and $k > i$, then we also add the intersection point $c_{i+1} := C_i \cap C_{i+1}$ to our collection. Thus, the collection of points

$$p_{ij} \in C_i, \quad 1 \leq j \leq s_i$$

is just the collection of locations on C_i in which the feathers with mother component C_i are born by a blowup. These points are called *base points* of the associated feathers. The collection $(p_{ij})_{1\le j\le s_i}$ defines a point Q_i in $\mathfrak{M}^+_{s_i}$.
Let now C_i be a $*$-component. Then we consider Q_i as a collection of points on $C_i\backslash(C_{i-1}\cup C_{i+1})$. Note that the intersection point c_{i+1} of C_i and C_{i+1} cannot belong to this collection due to Definition 2.27 (2) (ii). Identifying $C_i\backslash(C_{i-1}\cup C_{i+1})$ with $\mathbb{C}^*$ in a way that c_{i+1} corresponds to 0 and c_i to ∞ we obtain a point Q_i in the configuration space $\mathfrak{M}^*_{s_i}$.
Thus, in total, we obtain a point

$$Q(X,D) := (Q_2,\dots,Q_n) \in \mathfrak{M} = \mathfrak{M}^{\tau_2}_{s_2}\times\cdots\times\mathfrak{M}^{\tau_n}_{s_n},$$

where $\tau_i \in \{+,*\}$ represents the type of the corresponding component C_i. This point is called the *configuration invariant of* (X,D).
Performing elementary transformations in (X,D) with centers in C_0 does neither change Φ_0 nor the extended divisor (except of the weight C_1^2). Thus, it leaves the s_i and $Q(X,D)$ invariant. Therefore, we can define the configuration invariant for every m-standard completion of V.

If the boundary divisor has length n, then for a natural number t we let, for brevity,

$$t^\vee := n+2-t.$$

Proposition 2.29. *(Matching Principle, cf. [FKZ6], Prop. 3.3.1) Let $V = X\backslash D$ be a smooth Gizatullin surface completed by a standard zigzag D. Consider the reversed completion $(X^\vee, D^\vee)$ with boundary zigzag $D^\vee = C_0^\vee\cup\cdots\cup C_n^\vee$, associated numbers $s_2',\dots,s_n'$ and types $\tau_2',\dots,\tau_n'$. Then $s_i = s'_{i^\vee}$ and $\tau_i = \tau'_{i^\vee}$ for all $i = 2,\dots,n$. Moreover, the associated points $Q(X,D)$ and $Q(X^\vee,D^\vee)$ in $\mathfrak{M}$ coincide under the natural identification*

$$\mathfrak{M} = \mathfrak{M}^{\tau_2}_{s_2}\times\cdots\times\mathfrak{M}^{\tau_n}_{s_n} \cong \mathfrak{M}^{\tau_n'}_{s_n'}\times\cdots\times\mathfrak{M}^{\tau_2'}_{s_2'}.$$

Definition 2.30. *Let Γ and Γ' be weighted graphs. A* reconstruction *γ of Γ into Γ' is a finite sequence*

$$\gamma : \Gamma = \Gamma_0 \overset{\gamma_1}{\dashrightarrow} \Gamma_1 \overset{\gamma_2}{\dashrightarrow} \cdots \overset{\gamma_n}{\dashrightarrow} \Gamma_n = \Gamma',$$

where each arrow γ_i is either a blowup or a blowdown. The graph Γ' is called end graph *of γ. The inverse sequence $\gamma^{-1} := (\gamma_n^{-1},\dots,\gamma_1^{-1})$ yields a reconstruction of Γ' with end graph Γ.*
The reconstruction γ is said to be symmetric *if it is of the form (γ,γ^{-1}).*

Definition 2.31. *Two standard completions (X,D) and (X',D') of a Gizatullin surface V are* evenly linked *if there is a symmetric reconstruction of (X,D) into (X',D'). Otherwise they are called* oddly linked.

Indeed, a standard completion of a Gizatullin surface is always evenly linked to any other standard completion or to its inverse:

Lemma 2.32. *(cf. [FKZ6], Lemma 2.2.2) Let (X,D) and (X',D') be two standard completions of a Gizatullin surface $V \ncong \mathbb{A}^1\times\mathbb{C}^*$. After replacing, if necessary, (X,D) by its reversion $(X^\vee,D^\vee)$, (X,D) and (X',D') are evenly linked.*

The next theorem shows that the configuration invariant $Q(V)$ of a smooth Gizatullin surface V is indeed an invariant of the abstract isomorphism type of V:

Theorem 2.33. *(cf. [FKZ6], Theorem 3.4.1) Given two semi-standard completions $(X,D),(X',D')$ of smooth Gizatullin surface V, for the configuration invariants s_i, s_i' and $Q(X,D)\in\mathfrak{M}$, $Q(X',D')\in\mathfrak{M}'$ the following holds:*

1. *If (X,D) and (X',D') are evenly linked, then $s_i = s_i'$ for $i = 2,\dots,n$ and the points $Q(X,D)$ and $Q(X',D')$ of $\mathfrak{M} = \mathfrak{M}'$ coincide.*

(2) *If* (X,D) *and* (X',D') *are oddly linked, then* $s_i = s'_{i^\vee}$ *for* $i = 2,\ldots,n$ *and the points* $Q(X,D) \in \mathfrak{M}$ *and* $Q(X',D') \in \mathfrak{M}'$ *of* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *coincide under the natural identification*

$$\mathfrak{M} = \mathfrak{M}_{s_2}^{\tau_2} \times \cdots \times \mathfrak{M}_{s_n}^{\tau_n} \cong \mathfrak{M}_{s'_n}^{\tau'_n} \times \cdots \times \mathfrak{M}_{s'_2}^{\tau'_2} = \mathfrak{M}'.$$

Definition 2.34. *Given a configuration space* $\mathfrak{M} = \mathfrak{M}_{s_2}^{\tau_2} \times \cdots \times \mathfrak{M}_{s_n}^{\tau_n}$ *we consider the reversed product*

$$\mathfrak{M}^\vee := \mathfrak{M}_{s_n}^{\tau_n} \times \cdots \times \mathfrak{M}_{s_2}^{\tau_2}.$$

The symmetric configuration invariant *of a completion* (X,D) *of a smooth Gizatullin surface* V *is the unordered pair*

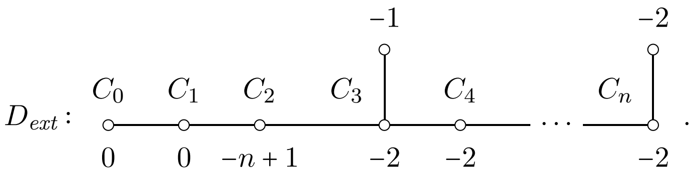
$$\tilde{Q}(X,D) := \{Q(X,D), Q(X^\vee,D^\vee)\}, \quad \textit{where} \quad Q(X,D) \in \mathfrak{M} \quad \textit{and} \quad Q(X^\vee,D^\vee) \in \mathfrak{M}^\vee.$$

Now, the following is obvious:

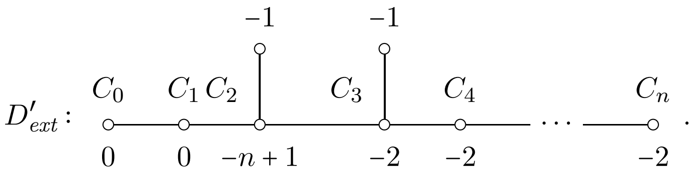
Corollary 2.35. *(cf. [FKZ6], Cor. 3.4.3) The pair* $\tilde{Q}(V) := \tilde{Q}(X,D)$ *as well as the sequence* $(s_2,\ldots,s_n)$ *(up to reversion) are invariants of the isomorphism type of* V.

2.4. **Distinguished and rigid extended divisors.** To motivate the rigidity of an extended divisor we consider the following example:

Examples 2.36. (1) *We consider a smooth Gizatullin surface with the extended divisor*

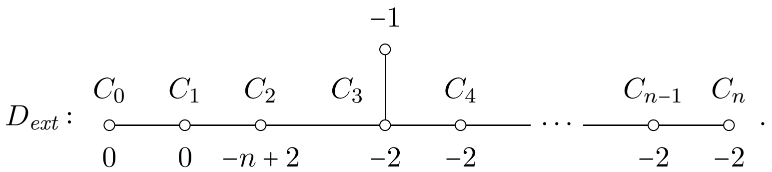


The mother component of the (-2)*-feather is* C_2*. But there also is a standard completion of the same surface with extended divisor*



Thus, the feather with mother component C_2 *can "jump".*

(2) *Now we consider a smooth Gizatullin surface with the following extended divisor:*



By the criterion given below we can show that the feather cannot jump to any other component. In this sense, this extended divisor is "rigid".

Definition 2.37. *The extended divisor* D_{ext} *is called* distinguished *if there is no index* i *with* $3 \le i \le n$*, such that* $D_{ext}^{>i}$ *is non-empty and contractible.*

We consider a family $(\mathcal{X},\mathcal{D})$ of standard completions of a Gizatullin surface V, parametrized by a smooth variety S (i. e. we have a flat morphism $\tau : \mathcal{X} \to S$). Thus, any $s \in S$ gives a standard completion $\mathcal{X}_s := \tau^{-1}(s)$ of V and therefore an extended divisor $\mathcal{D}_{\mathrm{ext}}(s)$. It is clear that the boundary divisor $\mathcal{D}(s)$ as well as the curves $\mathcal{R}_{i,j}(s)$ being resolutions of singularities of V stay constant. However, the feathers $\mathcal{F}_{i,j}(s)$ in general *depend* on s. This motivates the following definition:

Definition 2.38. *An extended divisor D_{ext} of a Gizatullin surface V is said to be* rigid *if for any flat family $\mathcal{X} \to S$ of standard completions of V with $\mathcal{D}_{ext}(s_0) = D_{ext}$ for a certain $s_0 \in S$ the dual graph of $\mathcal{D}_{ext}(s)$ is constant.*

For details on the notion of rigidity cf. [FKZ5], §2. Now, the following theorem gives a criterion for rigidity of an extended divisor, provided that it is distinguished.

Theorem 2.39. *(cf. [FKZ5], Theorem 2.17) An extended divisor D_{ext} is rigid, provided that all its bridge curves B_{ij} are (-1)-curves and one of the following conditions is satisfied:*

(1) $D_{ext}^{>n} \neq \varnothing$.
(2) *If for some i, $2 \le i < n$, the feather collection F_{i,j_i} is non-empty, then the divisor $D_{ext}^{\ge i+1}$ is not contractible.*

2.5. **Coordinates on smooth Gizatullin surfaces.** For our purpose we need explicit descriptions of smooth Gizatullin surfaces via affine coordinates on appropriate open affine charts. In [FKZ6], §4, such coordinates are constructed for the case of smooth Gizatullin surfaces which admit a *presentation*, i.e. where all boundary components C_i, $i \ge 2$ are +-components. We generalize this description for the case of arbitrary smooth Gizatullin surfaces. First, we recall this construction for the case of smooth Gizatullin surfaces which admit a presentation.

A standard completion (X, D) of a smooth Gizatullin surface V can be realized as a sequence of blowups of the quadric $Q = \mathbb{P}^1 \times \mathbb{P}^1$, such that all blowup centers are contained in $C_2 \backslash C_1$ and its infinitely near neighbourhood.
We consider on $X_1 = Q = \mathbb{P}^1 \times \mathbb{P}^1$ the affine chart $U_1 = Q \backslash (C_0 \cup C_1) \cong \mathbb{A}^2$ with affine coordinates $(x, y) = (x_1, y_1)$, such that $C_0 = \{y_1 = \infty\}$, $C_1 = \{x_1 = \infty\}$ and $C_2 = \{y_1 = 0\}$ holds and decompose the map $X \to \mathbb{P}^1 \times \mathbb{P}^1$ into single blowups

$$X = X_N \to X_{N-1} \to \cdots \to X_1 = Q.$$

We call a blowup $X_i \to X_{i-1}$ of *type (F)*, if it creates a feather F_i, otherwise we call it a blowup of *type (C)*. In the second case we let $F_i = \varnothing$. We start with the affine coordinates (x_1, y_1) on $U_1 = Q \backslash (C_0 \cup C_1)$ and construct recursively affine charts $U_i \cong \mathbb{A}^2$ with coordinates (x_i, y_i) on the intermediate surfaces X_i:

$(1)_i$ If C_s is the last curve of the zigzag constructed on X_i, we let

$$U_i = X_i \backslash \left(C_0 \cup C_1 \cup \cdots \cup C_{s-1} \cup \bigcup_{j \le i} F_j^\vee \right) \cong \mathbb{A}^2,$$

where $F_j^\vee = \varnothing$, if the blowup $X_j \to X_{j-1}$ is of type (C). Otherwise $F_j^\vee$ is the curve described explicitly below.

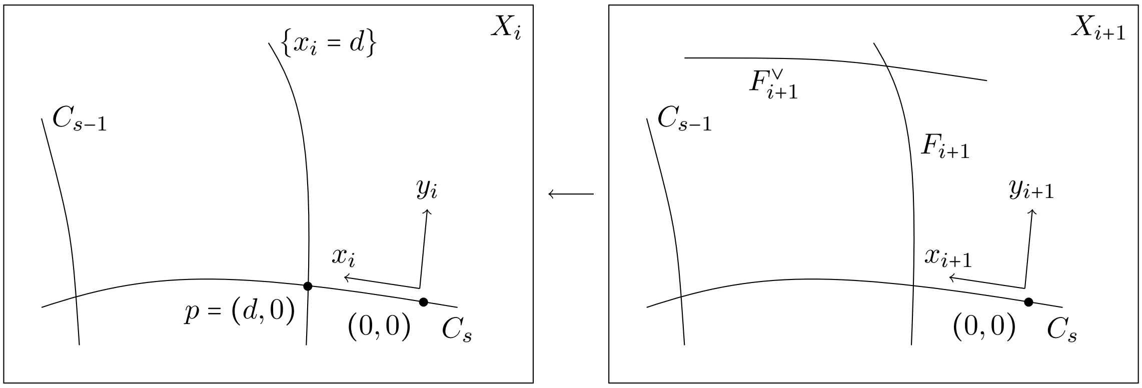
$(2)_i$ $C_s \cap U_i = \{y_i = 0\}$ and $x_i|_{C_s}$ is an affine coordinate on $C_s \backslash C_{s-1} \cong \mathbb{A}^1$.

$(3)_i$ If $X_i \to X_{i-1}$ is of type (F), then $F_i \cap U_i = \{x_i = 0\}$.

These properties obviously hold for U_1 and the coordinates (x_1, y_1) if we let $F_1^\vee = \varnothing$. Now we construct U_{i+1} and the coordinates (x_{i+1}, y_{i+1}) as follows:
Type (F): In the next blowup $X_{i+1} \to X_i$ with center $p = (d, 0) \in C_s$ a feather F_{i+1} is created. We let

$$(x_{i+1}, y_{i+1}) = \left(x_i, \frac{y_i}{x_i - d} \right) \quad \text{, such that} \quad (x_i, y_i) = (x_{i+1}, (x_{i+1} - d) y_{i+1}) \quad \text{holds.}$$

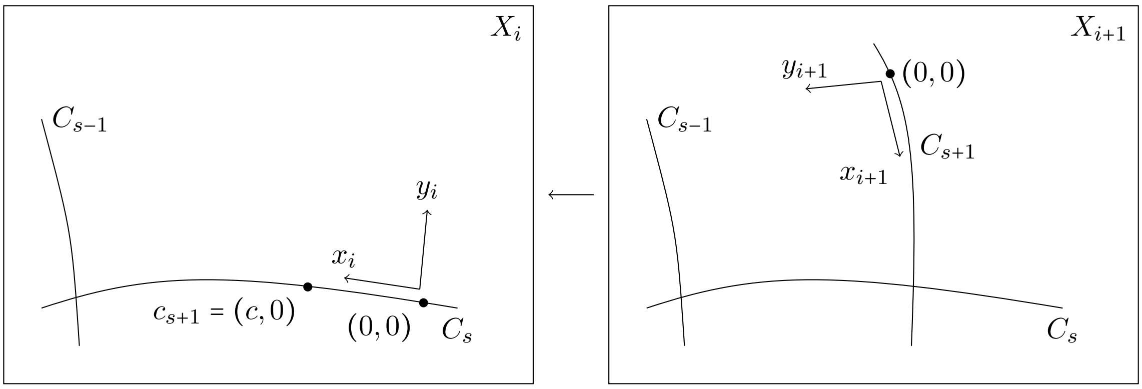
We let $F_{i+1}^\vee$ denote the proper transform on X_{i+1} (and on all further surfaces X_{i+j+1}) of the closure in X_i of the affine line $\{x_i = d\}$ (we intentionally use the symbol $^\vee$ since we will see that $(F_i, F_i^\vee)$ forms a pair of matching feathers for every non-empty F_i). It follows easily that the conditions $(1)_{i+1}$, $(2)_{i+1}$ and $(3)_{i+1}$ are satisfied for U_{i+1} and the coordinates (x_{i+1}, y_{i+1}).



Type (C): In the next blowup $X_{i+1} \to X_i$ with center $c_{s+1} = (c, 0) \in C_s$ a new component C_{s+1} is created. We let

$$(x_{i+1}, y_{i+1}) = \left(\frac{x_i - c}{y_i}, y_i\right) \quad \text{, such that} \quad (x_i, y_i) = (x_{i+1}y_{i+1} + c, y_{i+1}) \quad \text{holds.}$$

These are affine coordinates on the affine piece U_{i+1} as in $(1)_{i+1}$. The curve C_{s+1} is given on U_{i+1} by $\{y_{i+1} = 0\}$. Moreover, the rational function x_{i+1} has a first order pole along the curve C_s, hence $C_s \cap U_{i+1} = \varnothing$. Therefore, condition $(2)_{i+1}$ holds.



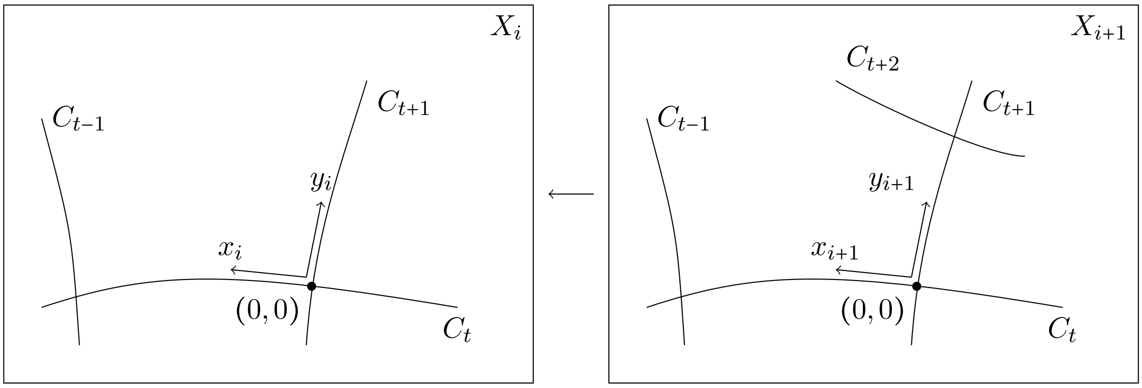
Indeed, $(F_i, F_i^\vee)$ is a pair of matching feathers:

Lemma 2.40. *(cf. [FKZ6], Lemma 5.2.2) If F_{i+1} and $F^\vee_{i+1}$ are non-empty then they form a pair of matching feathers on the surface $X = X_N$ as above.*

Remark 2.41. In our coordinate construction from above we only treated the case of outer blowups, i. e. where every boundary component C_i, $i \geq 2$, is a +-component. We consider once again a blowup of Type (F). If we introduce the coordinates (x_{i+1}, y_{i+1}) via

$$(x_i, y_i) = (x_{i+1}y_{i+1} + d, y_{i+1})$$

instead of $(x_i, y_i) = (x_{i+1}, (x_{i+1} - d)y_{i+1})$, then the equation of the feather F_{i+1} is given by $F_{i+1} \backslash D = \{y_{i+1} = 0\}$. In particular, the point $(x_{i+1}, y_{i+1}) = (0, 0)$ is the intersection point $F_{i+1} \cap F^\vee_{i+1}$.
This procedure can also be adopted for *inner* blowups. Therefore, if we create $\star$-components by inner blowups, we introduce affine coordinates inductively either via $(x_i, y_i) = (x_{i+1}y_{i+1}, y_{i+1})$ or via $(x_i, y_i) = (x_{i+1}, x_{i+1}y_{i+1})$. The only difference between blowups of type (C) and inner blowups is that in the last case *both* axes are given by certain boundary components, i. e. $\{y_i = 0\} = C_t \backslash C_{t-1}$ and $\{x_i = 0\} = C_{t+1} \backslash C_{t+2}$ for some t (cf. following figure for the case of the second transformation; in this figure the proper transform of C_{t+1} becomes C_{t+2}, since the exceptional curve of the blowup precedes the proper transform of C_{t+1}).



The description of the intersection points $F_{i+1} \cap F_{i+1}^\vee$ as well as the one of the correspondence fibration (cf. [FKZ6], 5.1, 5.2) remains valid, if we assume the following property (similarly to the case of +-components):

$(*)$ If F and G are two feathers with mother components C_s and C_t, $s \neq t$, then we create F before G if and only if $s < t$.

If we create a feather F_{i+1} with base point $(d,0) \in C_t \backslash (C_{t-1} \cup C_{t+1})$ then we introduce the coordinates (x_{i+1}, y_{i+1}) via $(x_i, y_i) = (x_{i+1}y_{i+1} + d, y_{i+1})$ as mentioned above. Similarly, we introduce the coordinates (x_{i+1}, y_{i+1}) via $(x_i, y_i) = (x_{i+1}, x_{i+1}y_{i+1} + d)$ if we want to create a feather F_{i+1} with base point $(0,d) \in C_t \backslash (C_{t-1} \cup C_{t+1})$. It is easy to see that in both cases the point $(x_{i+1}, y_{i+1}) = (0,0) = p$ is the intersection point $p = F_{i+1} \cap F_{i+1}^\vee$.

3. Gizatullin surfaces with a distinguished and rigid extended divisor

Notation: According to Lemma 1.0.7 in [BD], any smooth 1-standard pair (X,D) may be obtained by some blowups of points on a fiber of $\mathbb{F}_1$. An embedding of $\mathbb{F}_1$ into $\mathbb{P}^2 \times \mathbb{P}^1$ is given by

$$\mathbb{F}_1 = \{((x:y:z),(s:t)) \in \mathbb{P}^2 \times \mathbb{P}^1 \mid yt - zs = 0\}.$$

In other words, $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ in $(1:0:0)$. We denote by $\tau : \mathbb{F}_1 \to \mathbb{P}^2$ the projection onto the first factor and by C_0, L the lines $\{z = 0\}$ and $\{y = 0\}$ respectively. We also denote by C_0 and L their proper transforms on $\mathbb{F}_1$, by C_1 the exceptional curve $\tau^{-1}(1:0:0) = \{(1:0:0)\} \times \mathbb{P}^1$ and by L_0 the affine line $L \backslash C_1 \subseteq \mathbb{F}_1$ as well as its image $L \backslash \{(1:0:0)\}$ in $\mathbb{P}^2$. Moreover, we have isomorphisms

$$\mathbb{A}^2 \xrightarrow{\cong} \mathbb{F}_1 \backslash (C_0 \cup C_1), \quad (x_0, y_0) \mapsto ((x_0 : y_0 : 1), (y_0 : 1))$$

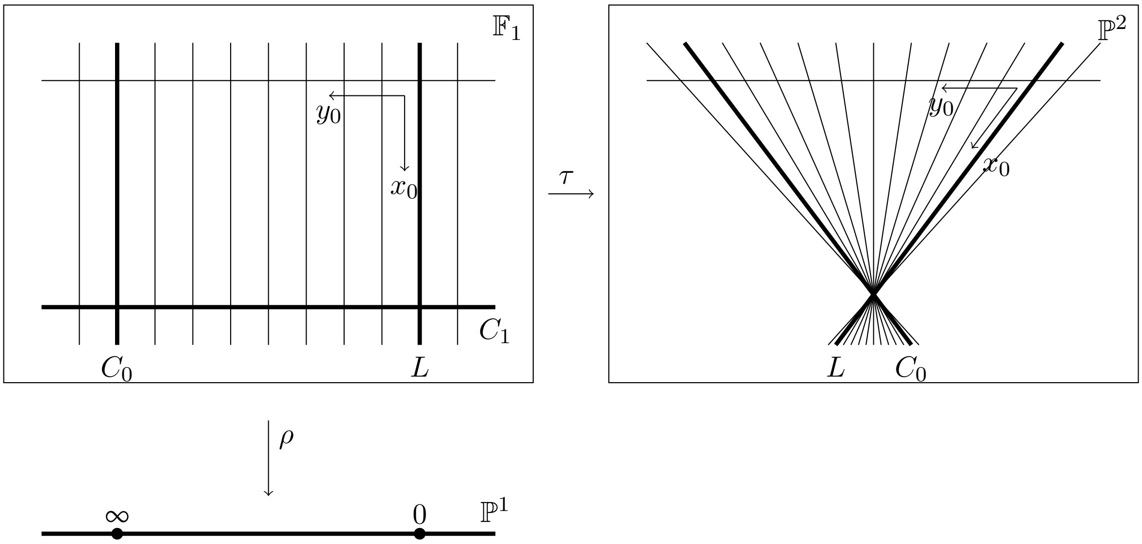
as well as

$$\mathbb{A}^2 \xrightarrow{\cong} \mathbb{P}^2 \backslash C_0, \quad (x_0, y_0) \mapsto (x_0 : y_0 : 1).$$

In these coordinates the affine line L_0 is given by $y_0 = 0$. In the following we will always denote these coordinates on $\mathbb{F}_1 \backslash (C_0 \cup C_1)$ by (x_0, y_0). $\mathbb{F}_1$ admits a $\mathbb{P}^1$-fibration

$$\rho : \mathbb{F}_1 \to \mathbb{P}^1, \quad ((x:y:z),(s:t)) \mapsto (s:t),$$

which is just the projection on $C_1 \cong \mathbb{P}^1$. The restriction of ρ to $\mathbb{A}^2 = \mathbb{F}_1 \backslash (C_0 \cup C_1)$ yields an $\mathbb{A}^1$-fibration $\pi : \mathbb{A}^2 \to \mathbb{A}^1$, which is simply the projection onto the second factor $(x_0, y_0) \mapsto y_0$.



We denote by **Aff** the group of automorphisms of $\mathbb{A}^2$, which extend to automorphisms of $\mathbb{P}^2$ and by **Jon** the group of triangular (de Jonquieres) automorphisms (automorphisms of $(\mathbb{A}^2, \pi)$). In other words, we have

$$\begin{array}{rcl} \mathbf{Aff} & = & \{(x_0, y_0) \mapsto (a_{11}x_0 + a_{12}y_0 + b_1, a_{21}x_0 + a_{22}y_0 + b_2) \mid a_{11}a_{22} - a_{12}a_{21} \neq 0\} \\ \mathbf{Jon} & = & \{(x_0, y_0) \mapsto (ax_0 + P(y_0), by_0 + c) \mid a, b \in \mathbb{C}^*, c \in \mathbb{C}, P(y_0) \in \mathbb{C}[y_0]\}. \end{array}$$

Moreover, if we consider a reversion $(X, D) \dashrightarrow (X', D')$ of 1-standard pairs centered in a point $p \in C_0 \backslash C_1$, we associate this point with its image $(\lambda : 1 : 0)$ in $\mathbb{P}^2$ via the map $\tau \circ \eta : X \to \mathbb{P}^2$.

We remember the following lemma, which is an important tool to compute the graph $\mathcal{F}_V$ explicitly (cf. [BD], Lemma 5.2.1):

Lemma 3.1. *For $i = 1, 2$, let $(X_i, D_i, \overline{\pi}_i)$ be a 1-standard pair with a minimal resolution of singularities $\mu_i : (Y_i, D_i, \overline{\pi}_i \circ \mu_i) \to (X_i, D_i, \overline{\pi}_i)$ and let $\eta_i : Y_i \to \mathbb{F}_1$ be the (unique) birational morphism. Then, the following statements are equivalent:*

(a) *The $\mathbb{A}^1$-fibered surfaces $(X_1 \backslash D_1, \pi_1)$ and $(X_2 \backslash D_2, \pi_2)$ (respectively the pairs $(X_1, D_1, \overline{\pi}_1)$ and $(X_2, D_2, \overline{\pi}_2)$) are isomorphic.*

(b) *There exists an element of **Jon** (respectively of **Jon**$\cap$**Aff**) which sends the points blown-up by η_1 onto those blown-up by η_2 and sends the curves contracted by μ_1 onto those contracted by μ_2.*

3.1. **Smooth Gizatullin surfaces with a distinguished and rigid extended divisor.** There is a good chance to obtain "nice" $\mathbb{A}^1$-fibered affine surfaces, if the extended divisor is distinguished and rigid. Before considering some examples, we remember the following:
Blowing up the plane $\mathbb{A}^2$ in the point $(0,0)$ we obtain the quasi-projectice variety $X = \{((x,y),(s:t)) \mid xt - ys = 0\} \subseteq \mathbb{A}^2 \times \mathbb{P}^1$ with the map $\pi : X \to \mathbb{A}^2$, $((x,y),(s:t)) \mapsto (x,y)$, which can be covered by two affine charts X_1 and X_2, both isomorphic to $\mathbb{A}^2$. Precisely, if we denote by $C_1 = \{y = 0\}$ and $C_2 = \{x = 0\}$ the axes and by E the exceptional curve, then

$$X_1 = X \backslash \hat{C}_1, \quad X_2 = X \backslash \hat{C}_2.$$

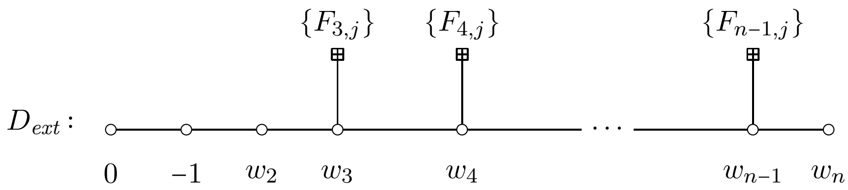
Introducing the coordinates $(x_1, y_1) = (x, t/s)$ on X_1 and $(x_2, y_2) = (s/t, y)$ on X_2, the restriction of the map π on these affine charts is given by

$$\pi : X_1 \to \mathbb{A}^2, (x_1, y_1) \mapsto (x_1, x_1 y_1) \text{ and } \pi : X_2 \to \mathbb{A}^2, (x_2, y_2) \mapsto (x_2 y_2, y_2)$$

respectively. We use these maps in our descriptions.

In the following we consider smooth Gizatullin surfaces with a distinguished and rigid extended divisor.

Theorem 3.2. *Let V be a smooth Gizatullin surface which admits a distinguished and rigid extended divisor with $n \geq 4$. Then the dual graph of D_{ext} has the form*



with $1 \leq j \leq r_i$. Moreover, the following hold:

(1) *For any two 1-standard completions (X', D') and (X'', D'') of V with $(X'\backslash D', \pi') \cong (X''\backslash D'', \pi'')$ we have $(X', D', \bar{\pi}') \cong (X'', D'', \bar{\pi}'')$.*
(2) *The graph $\mathcal{F}_V$ has one of the following two forms:*

$$\mathcal{F}_V : \ [(X, D)] \bullet \longleftrightarrow \bullet \ [(X^\vee, D^\vee)] \quad \text{or} \quad \mathcal{F}_V : [(X, D)] \bullet \circlearrowleft .$$

If $\mathcal{F}_V$ is of the form $\bullet \circlearrowleft$, then $D^{\geq 2}$ is a palindrome and $r_i = r_{i^\vee}$ and $Q_i = Q_{i^\vee}$ holds for all $i = 3, \ldots, n-1$.
(3) *If $r_i > 0$ holds for at most two indices $i \in \{3, \ldots, n-1\}$, then $\mathcal{F}_V$ has the form $\bullet \circlearrowleft$ if and only if $D^{\geq 2}$ is a palindrome and $r_i = r_{i^\vee}$ and $Q_i = Q_{i^\vee}$ holds for all $i = 3, \ldots, n-1$.*
(4) *$\mathrm{Aut}(V)$ is generated by automorphisms of $\mathbb{A}^1$-fibrations if and only if $\mathcal{F}_V$ has no loops except for the case $\Gamma_D = [[0, -1, -2, -2, -2]]$.*

Proof. First, notice that a distinguished and rigid extended divisor of a *smooth* Gizatullin surface cannot admit feathers which are attached to the component C_2 or to C_n. Thus the extended divisor has the desired form. Moreover, the rigidity of D_{ext} implies that all feathers are (-1)-feathers and are therefore attached to their mother components.
Since D_{ext} is distinguished and rigid, the extended divisor $D^\vee_{\text{ext}}$ has the same property. Thus the components $C_3, \ldots, C_{n-1}$ are $*$-components in any 1-standard completion of V. We consider two 1-standard completions (X', D') and (X'', D'') of V such that $(X'\backslash D', \pi') \cong (X''\backslash D'', \pi'')$. By Lemma 3.1 an isomorphism $\mathbf{Jon} \ni \psi : (X'\backslash D', \pi') \xrightarrow{\sim} (X''\backslash D'', \pi'')$ sends the centers of the blowup $\eta' : X' \to \mathbb{F}_1$ onto those of $\eta'' : X'' \to \mathbb{F}_1$. We present ψ in the coordinates (x_0, y_0) of $\mathbb{A}^2 = \mathbb{F}_1 \backslash (C_0 \cup C_1)$ as

$$\psi(x_0, y_0) = (ax_0 + P(y_0), by_0 + c), \quad a, b \in \mathbb{C}^*, c \in \mathbb{C}, P(y_0) \in \mathbb{C}[y_0],$$

cf. Lemma 2.12. Assuming that the blowup process for both surfaces starts at $(0,0) \in L_0$ we obtain $c = P(0) = 0$ and φ has the form

$$\psi(x_0, y_0) = (ax_0 + y_0 Q(y_0), by_0), \quad a, b \in \mathbb{C}^*, Q(y_0) \in \mathbb{C}[y_0].$$

We decompose η' into single blowups $X' = X'_{n-2+m} \to \cdots \to X'_{n-2} \to \cdots \to X'_0 = \mathbb{F}_1$, such that we create the boundary components by inner blowups in $X'_{n-2} \to \cdots \to X'_0 = \mathbb{F}_1$ and the $m = r_3 + \cdots + r_{n-1}$ feathers in $X'_{n-2+m} \to \cdots \to X'_{n-2}$ (similarly for X''). We fix an index $j \in \{3, \ldots, n-1\}$ and consider ψ on the component C_j on the surface X_{n-2}. In each step we introduce inductively affine coordinates (x_i, y_i) on X_i, such that:

(1) $C_j \backslash C_{j-1} = \{y_{n-2} = 0\}$ and $C_{j+1} \backslash C_{j+2} = \{x_{n-2} = 0\}$ on X_{n-2} (here we define C_{j+2} to be the infinity point if $j = n-1$),
(2) after each blowup we have either $(x_i, y_i) = (x_{i+1}, y_{i+1})$ or $(x_i, y_i) = (x_{i+1}, x_{i+1} y_{i+1})$ or $(x_i, y_i) = (x_{i+1} y_{i+1}, y_{i+1})$.

We refer to the last two coordinate transformations as to transformations of type 1 and type 2 respectively. Denoting the lifting of ψ on the surface X_i by Ψ_i, we show by induction that Ψ_{n-2} has the form

$$\Psi_{n-2}(x_{n-2},y_{n-2})=\left(\alpha x_{n-2}(1+x_{n-2}^k y_{n-2}^l R(x_{n-2}^s y_{n-2}^t))^p,\frac{\beta y_{n-2}}{(1+x_{n-2}^k y_{n-2}^l R(x_{n-2}^s y_{n-2}^t))^q}\right),$$

such that $R=\gamma\cdot Q$ with an appropriate $\gamma\in\mathbb{C}^*$, $k\geq 0$, $l,s,t,p,q\geq 1$ and α and β are monomials in a and b. In the first step we blow up $\mathbb{F}_1$ in the point $(0,0)\in L_0$ and introduce coordinates $(x_1,y_1)=(x_0,x_0y_0)$. This leads to

$$\Psi_1(x_1,y_1)=\left(x_0(a+y_0Q(x_0y_0)),\frac{by_0}{a+y_0Q(x_0y_0)}\right)=\left(ax_0(1+y_0R(x_0y_0)),\frac{(b/a)y_0}{1+y_0R(x_0y_0)}\right)$$

with $R=\frac{1}{a}Q$. In the induction step we consider the case $(x_i,y_i)=(x_{i+1},x_{i+1}y_{i+1})$. A short computation yields

$$\Psi_{i+1}(x_{i+1},y_{i+1})=\left(\alpha x_{i+1}(1+x_{i+1}^{k+l}y_{i+1}^l R(x_{i+1}^{s+t}y_{i+1}^t))^p,\frac{(\beta/\alpha)y_{i+1}}{(1+x_{i+1}^{k+l}y_{i+1}^l R(x_{i+1}^{s+t}y_{i+1}^t))^{p+q}}\right).$$

Similarly we obtain in the case $(x_i,y_i)=(x_{i+1}y_{i+1},y_{i+1})$

$$\Psi_{i+1}(x_{i+1},y_{i+1})=\left((\alpha/\beta)x_{i+1}(1+x_{i+1}^k y_{i+1}^{k+l}R(x_{i+1}^s y_{i+1}^{s+t}))^{p+q},\frac{\beta y_{i+1}}{(1+x_{i+1}^k y_{i+1}^{k+l}R(x_{i+1}^s y_{i+1}^{s+t}))^q}\right).$$

The case $(x_i,y_i)=(x_{i+1},y_{i+1})$ is obvious (we need such transformations only when we perform inner blowups centered outside of the affine piece with coordinates (x_i,y_i)). Moreover, we see from the induction step that $k=0$ holds if and only if we perform no blowups of type 1 except for the first one in $(0,0)\in L_0$.
Thus Ψ_{n-2} induces on $C_j\backslash(C_{j-1}\cup C_{j+1})\cong\mathbb{C}^*$ the map

$$\Psi_{n-2}(x,0)=(\alpha x,0).$$

But the affine map

$$\tilde{\psi}(x_0,y_0):=(ax_0+cy_0,by_0)\quad\text{with}\quad c=Q(0),$$

defines the same map on $C_j\backslash(C_{j-1}\cup C_{j+1})$. In particular, $\tilde{\psi}$ sends the points blown-up by $\eta':X'\to\mathbb{F}_1$ onto those blown-up by $\eta'':X''\to\mathbb{F}_1$. By Lemma 3.1 we obtain that $(X',D',\bar{\pi}')\cong(X'',D'',\bar{\pi}'')$. This shows (1).
Now we show that the graph $\mathcal{F}_V$ admits only one arrow (the first part of assertion (2) follows immediately since $\mathcal{F}_V$ is connected). Due to (1) we can choose $(X,D,\bar{\pi})$ itself as a representative of the conjugacy class $[(X,D)]\in\mathcal{F}_V$. The automorphism

$$\psi_a(x_0,y_0)=(x_0+ay_0,y_0),\quad a\in\mathbb{C},$$

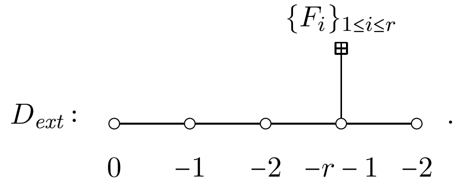
of $\mathbb{A}^2=\mathbb{F}_1\backslash(C_0\cup C_1)$ can be lifted to an automorphism of (X,D). Moreover, ψ_a induces on $C_0\backslash C_1\cong\mathbb{A}^1$ the translation $\lambda\mapsto\lambda+a$. Thus $\mathrm{Aut}(X,D)$ acts transitively on $C_0\backslash C_1$ and there is only one arrow starting from $[(X,D)]$.
The condition on $\mathcal{F}_V$ to have a loop is equivalent to $(X,D,\bar{\pi})\cong(X^\vee,D^\vee,\bar{\pi}^\vee)$. This implies the second part of (2).
To show assertion (3), we consider two 1-standard completions (X',D') and (X'',D'') of V such that there exist two indices $s,t\in\{3,\dots,n-1\}$, $s<t$, with $r_i',r_i''>0$ only for $i\in\{s,t\}$ and $(Q_s',Q_t')=(Q_s'',Q_t'')$. It is sufficient to show that this implies $(X',D',\bar{\pi}')\cong(X'',D'',\bar{\pi}'')$. Then we get the non-trivial direction of (3) as follows:
If $(X,D,\bar{\pi})$ is a 1-standard completion of V with $r_i=r_{i^\vee}>0$, $r_j=0$ for $j\neq i,i^\vee$ and $Q_i=Q_{i^\vee}$,

the Matching Principle yields $r^\vee_{i^\vee} = r_i$, $r^\vee_i = r_{i^\vee}$, $Q^\vee_{i^\vee} = Q_i$ and $Q^\vee_i = Q_{i^\vee}$ for the corresponding invariants. Thus we obtain $r^\vee_k = r_k$ and $Q^\vee_k = Q_k$, $k \in \{i, i^\vee\}$ and $(X, D, \bar{\pi}) \cong (X^\vee, D^\vee, \bar{\pi}^\vee)$ follows. Thus the graph $\mathcal{F}_V$ admits only one vertex.
Again we consider the blowup process $(X'_{n-2+m}, D'_{n-2+m}) \to \cdots \to (X'_{n-2}, D'_{n-2}) \to \cdots \to (X'_0, D'_0) = (\mathbb{F}_1, C_0 \cup C_1)$, where we create the boundary components in $(X'_{n-2}, D'_{n-2}) \to \cdots \to (X'_0, D'_0) = (\mathbb{F}_1, C_0 \cup C_1)$ and the $m = r'_s + r'_t$ feathers in $(X'_{n-2+m}, D'_{n-2+m}) \to \cdots \to (X'_{n-2}, D'_{n-2})$. The 2-torus action on $\mathbb{F}_1 \backslash (C_0 \cup C_1)$ lifts to (X'_{n-2}, D'_{n-2}), inducing maps $x \mapsto \alpha_i x$ on $C_i \backslash (C_{i-1} \cup C_{i+1}) \cong \mathbb{C}^*$ (we identify $C_i \backslash (C_{i-1} \cup C_{i+1})$ with $\mathbb{C}^*$ in a way that $C_i \cap C_{i-1} = \{\infty\}$ and $C_i \cap C_{i+1} = \{0\}$). Ignoring these maps on C_i for $i \neq s, t$, the maps $x \mapsto \alpha_s x$ and $x \mapsto \alpha_t x$ can be chosen arbitrarily. Thus denoting by A'_s, A'_t, A''_s and A''_t the corresponding base point sets of the feathers, there exists a 2-torus action μ such that $\mu.A'_s = A''_s$ and $\mu.A'_t = A''_t$. Therefore we get $(X', D') \cong (X'', D'')$.
To show (4), we let $\Gamma_D \neq [[0, -1, (-2)_3]]$. It is easy to see that this condition is equivalent to the existence of a boundary component of self-intersection ≤ -3. In this case assertion (4) is a direct consequence of Proposition 2.19 and assertion (2).
It remains the case where $\Gamma_D = [[0, -1, (-2)_3]]$. We choose a fixed reversion $\Psi_0 : (X, D) \dashrightarrow (X, D)$ with center $p \in C_0 \backslash C_1$ and such that p is also the center of Ψ_0^{-1} (we can always choose such a reversion since $\mathrm{Aut}(X, D)$ acts transitively on $C_0 \backslash C_1$). Let $\alpha \in \mathrm{Aut}(X, D)$ be an element, which does not fix the center of Ψ_0. Then the reversions $\Psi_0^{-1} = \Psi_0$ and $\Psi_0 \alpha$ have distinct base points, i.e. $\Psi_0 \alpha \Psi_0$ is a reversion (cf. Lemma 2.15), equal to $\beta \Psi_0 \gamma$ for some $\beta, \gamma \in \mathrm{Aut}(X, D)$. Therefore we have $\Psi_0 = (\alpha^{-1} \Psi_0)(\beta \Psi_0 \gamma) = \alpha^{-1}(\Psi_0 \beta \Psi_0^{-1})\gamma$. Since β preserves the fibration π, there is a $\varphi \in \mathrm{Aut}(\mathbb{A}^1)$ such that $\pi\beta = \varphi\pi$. This yields $(\pi\Psi_0)(\Psi_0 \beta \Psi_0^{-1}) = \varphi(\pi\Psi_0)$, i.e. the map $\Psi_0 \beta \Psi_0^{-1}$ is compatible with the fibration $\pi\Psi_0$. Thus the reversion Ψ_0 is generated by automorphisms of $\mathbb{A}^1$-fibrations. In Corollary 3.14 we show that $\mathrm{Aut}(V)$ is generated by $\langle \mathrm{Aut}(X, D), \Psi_0 \rangle$ and $\mathrm{Aut}(V, \pi)$ (the elements of $\mathrm{Aut}(X, D)$ as well as Ψ_0 give automorphisms of V by restriction). It follows that $\mathrm{Aut}(V)$ is generated by automorphisms of $\mathbb{A}^1$-fibrations. □

In the spacial case of Theorem 3.2 with only one family of (-1)-feathers one can show that the extended divisor is automatically distinguished and rigid. We can even extract some additional informations:

Corollary 3.3. *Let V be as in Theorem 3.2 such that $r_i > 0$ holds for exactly one index $i = j \in \{3, \ldots, n-1\}$. Then the following hold:*

(1) *$D^{\geq 2}$ is a palindrome if and only if $\Gamma_D = [[-2, -r-1, -2]]$ with $r \geq 1$. In this case, D_{ext} has the form*



In particular, $\mathrm{Aut}(V)$ is generated by automorphisms of $\mathbb{A}^1$-fibrations if and only if

$$\Gamma_D \neq [[0, -1, -2, -r-1, -2]], \quad r \geq 2.$$

(2) *V admits a $\mathbb{C}^*$-action and the DPD-representation is given by the following data:*

$$C = \mathbb{A}^1, \quad D_+ = -\frac{e}{m}[0], \quad D_- = \frac{e}{m}[0] - \sum_{i=1}^{r} [a_i],$$

where $\frac{m}{e} = [-w_{j+1}, \ldots, -w_n]$ and $a_i \in \mathbb{C}^* \cong C_i \backslash (C_{i-1} \cup C_{i+1})$ are the base points of the feathers. Conversely, for each pair (e, m) with $0 \leq e < m$ and $\gcd(e, m) = 1$ there exists a smooth Gizatullin surface with such an extended divisor.

Proof. Assertion (1) is an easy exercise and assertion (2) follows directly from [FKZ4], Corollary 5.10. For the notion of a DPD-representation of a surface with an effective $\mathbb{C}^*$-action see [FKZ1]. □

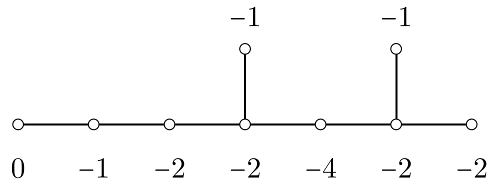
Remark 3.4. (1) Theorem 3.2 yields in particular that every $\mathbb{A}^1$-fibration $\varphi : V \to \mathbb{A}^1$ on such a surface V is conjugated either to $\Phi_0 := \Phi_{|C_0|} : V \to \mathbb{A}^1$ or to $\Phi_0^\vee := \Phi_{|C_0^\vee|} : V \to \mathbb{A}^1$. This is a special case of Theorem 5.10 in [FKZ5], where the same assertion is shown for normal Gizatullin surfaces with a distinguished and rigid extended divisor.
(2) We can show even more. It is not hard to see that the following conditions are even equivalent:

(1) $\mathcal{F}_V$ has one of the following forms:

$$\mathcal{F}_V : \bullet \longleftrightarrow \bullet \qquad \text{or} \qquad \mathcal{F}_V : \bullet \circlearrowleft .$$

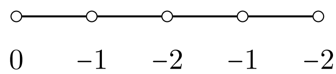
(2) If (X, D) is a 1-standard completion of V, then the extended divisor D_{ext} is distinguished and rigid.

However, the weaker condition that V admits at most two conjugacy classes of $\mathbb{A}^1$-fibrations does *not* imply that the extended divisor of V is distinguished and rigid. We consider the following example: Let V be a smooth Gizatullin surface admitting a 1-standard completion with extended divisor

D_{ext} : 

.

The components C_2, C_4 and C_6 are +-components, while C_3 and C_5 are of type $*$. Thus this graph is indeed rigid (feathers attached to $*$-components cannot jump), but not distinguished. We can construct (X, D) from $(\mathbb{F}_1, C_0 \cup C_1)$ in the following way: we blowup two times in $(0,0) \in L_0$ and introduce coordinates (x_1, y_1) via $(x_0, y_0) = (x_1, x_1^2 y_1)$ as well as coordinates (u_1, v_1) via $(x_0, y_0) = (u_1 v_1, v_1)$. This leads to the dual graph



.

Now we perform an outer blowup in a point $P = (\beta, 0) \in C_4 \backslash C_3$ (in coordinates (u_1, v_1)), obtaining the component C_6. Finally we perform an inner blowup in $C_4 \cap C_6$, which results in the component C_5. After the blowup in P we introduce the coordinates (u_2, v_2) via $(u_1 - \beta, v_1) = (u_2, u_2 v_2)$ and after the last blowup the coordinates (u_3, v_3) via $(u_2, v_2) = (u_3, u_3 v_3)$. We denote the resulting surface by $(\tilde{X}, \tilde{D})$. In the last step we create both (-1)-feathers by blowing up in some points $Q_1 = (0, \alpha) \in C_3 \backslash (C_2 \cup C_4)$ (in coordinates (x_1, y_1)) and $Q_2 = (0, \gamma) \in C_5 \backslash (C_4 \cup C_6)$ (in coordinates (u_3, u_3)). An appropriate automorphism $\psi(x_0, y_0) = (a x_0 + b y_0, c y_0)$ of $(\mathbb{F}_1, C_0 \cup C_1)$ can bring the surface (X, D) in a "standard form": The condition $b = -a\beta$ moves the point $(\beta, 0)$ on $(0,0)$ (in coordinates (u_1, v_1)). Thus we may assume that $\beta = 0$. Then ψ can be lifted to $(\tilde{X}, \tilde{D})$ if and only if $b = 0$ holds and the lifting Ψ of ψ has the following forms in the coordinates introduced above:

$$\Psi(x_1, y_1) = (a x_1, a^{-2} c y_1) \quad \text{and} \quad \Psi(u_3, v_3) = (a c^{-1} u_3, a^{-2} c^3)$$

The conditions $\Psi(Q_1) = (0,1)$ and $\Psi(Q_2) = (0,1)$ (in the corresponding affine coordinates) leads to $c = a^2$ and $c^2 = 1$, or equivalently $(a, c) \in \{(1,1), (-1,1), (i,-1), (-i,-1)\}$. In particular, ac^{-1} can take every value in $W_4 = \{z \in \mathbb{C}^* \mid z^4 = 1\} = \{\pm 1, \pm \mathrm{i}\}$. Therefore we obtain:

(1) V does not depend on any parameter.
(2) $(X, D) \cong (X^\vee, D^\vee)$.
(3) For two 1-standard completions (X, D) and (X', D') of V it follows $(X, D) \cong (X', D')$ (cf. Lemma 3.1). In particular, V admits only one conjugacy class of $\mathbb{A}^1$-fibrations.

The subgroup $\mathrm{Aut}(X, D)$ does *not* act transitively on $C_0 \backslash C_1$. If we identify $C_0 \backslash C_1$ as always with $\mathbb{A}^1$ via $(\lambda : 1 : 0) \mapsto \lambda$, then the orbit of a point $\lambda \in C_0 \backslash C_1$ is given by $\mathrm{Aut}(X, D).\lambda = \lambda \cdot W_4$ ($\psi(x_0, y_0) = (a x_0, c y_0)$ induces on $C_0 \backslash C_1$ the map $\lambda \mapsto ac^{-1} \cdot \lambda$). Thus the graph $\mathcal{F}_V$ has the form

$$\mathcal{F}_V : \quad \lambda_1 \circlearrowleft \overset{\cdots}{\bullet} \circlearrowright \lambda_2 \quad ,$$

where the arrows are in a $1:1$-correspondence to elements of $\mathbb{C}/W_4$.

It is easy to see that for smooth Gizatullin surfaces with a distinguished and rigid extended divisor the divisor $D^{\geq 2}$ as well as the number r_i of feathers attached to C_i cannot be symmetric simultaneously (i. e. $r_i = r_{i^\vee}$) if n is odd. Thus we obtain the following

Corollary 3.5. *Let V be as in Theorem 3.2 and let n be odd. Then* $\mathrm{Aut}(V)$ *is generated by automorphisms of $\mathbb{A}^1$-fibrations.*

3.2. **The orbit decomposition of the action of** $\mathrm{Aut}(V)$**.** Our next goal is to describe the action of the automorphism group of such surfaces. We will show that these surfaces in general admit points, which do not belong to the big orbit O of $\mathrm{Aut}(V)$. These points are even fix points of $\mathrm{Aut}(V)$, if the base points of the feathers are "in general position". We start with the following simple observation:

Proposition 3.6. *Let V be a smooth Gizatullin surface with a 1-standard completion $(X, D = C_0 \rhd C_1 \rhd \cdots \rhd C_n)$ and the associated $\mathbb{P}^1$-fibration $\overline{\pi} : X \to \mathbb{P}^1$. Moreover, let F_i be the feathers of the extended divisor D_{ext} and let $F_i^\vee$ be the matching feathers in $D_{ext}^\vee$, where $(X^\vee, D^\vee) \dashrightarrow (X, D)$ is a reversion with an arbitrary center $p \in C_0 \backslash C_1$. Denoting by O the big orbit of the natural action of the automorphism group of V, we have*

$$V \backslash O \subseteq \bigcup_i F_i \cap F_i^\vee .$$

In particular, the fix point set $F(V) := \{x \in V \mid \mathrm{Aut}(V).x = x\}$ is finite and contained in $\bigcup_i F_i \cap F_i^\vee$.

Proof. Since $\mathrm{Aut}(V)$ acts on V with a big orbit O, each fiber F' of $\pi = \overline{\pi}|_V$ intersects O. Since $F_0 := \overline{\pi}^{-1}(\overline{\pi}(C_2 \cup \cdots C_n))$ is the only degenerated fiber of $\overline{\pi}$, every point of $V \backslash O$ cannot belong to any other fiber F. Indeed, if $D = C_0 \cup \cdots \cup C_n$, then the automorphism $\psi(x_0, y_0) = (x_0 + y_0^{n-1}, y_0)$ of $\mathbb{A}^2 = \mathbb{F}_1 \backslash (C_0 \cup C_1)$ can be lifted to an automorphism of V, which induces a non-trivial translation $x \mapsto x + y^{n-1}$ on any other fiber $F = \pi^{-1}(y) \cong \mathbb{A}^1$, $y \neq 0$ (this can be checked by using our affine coordinates introduced in Section 2.5). Thus $V \backslash O \subseteq F_0 \cap V$.
Let now $x \in V$ be a point contained in

$$\left(\bigcup_i F_i \right) \backslash \left(\bigcup_i F_i \cap F_i^\vee \right).$$

Then x is contained in some regular fiber of the $\mathbb{A}^1$-fibration $\pi^\vee$, thus it can be moved continuously by an appropriate $\mathbb{C}_+$-action. Therefore $x \in O$. In a similar way we obtain that any point in

$$\left(\bigcup_i F_i^\vee \right) \backslash \left(\bigcup_i F_i \cap F_i^\vee \right)$$

can be moved continuously by automophisms of V. It follows that $V \backslash O \subseteq \bigcup_i F_i \cap F_i^\vee$. □

Considering the simplest examples of smooth Gizatullin surfaces like the affine plane $\mathbb{A}^2$, the Danielewski surfaces $V_P = \{xy - P(z) = 0\} \subseteq \mathbb{A}^3$, $P(z) \in \mathbb{C}[z]$ with simple roots, or the Danilov-Gizatullin surfaces V_{k+1} one sees easily that the automorphism group acts transitively on these surfaces. Moreover, Gizatullin conjectured in [Gi] II that for smooth Gizatullin surfaces the action of the automorphism group is always transitive, i. e. that V coincides with the big orbit O of $\mathrm{Aut}(V)$:

Conjecture (Gizatullin): Let V be a smooth Gizatullin surface. Then the action of the automorphism group is always transitive on V, i. e. V coincides with the big orbit O of $\mathrm{Aut}(V)$.

The main result of this paper is a class of smooth Gizatullin surfaces which yield counterexamples to Gizatullin's conjecture. Indeed, we show that the automorphism group of surfaces considered in Theorem 3.2 does not act transitively in general. Preliminary, we need the following simple lemma:

Lemma 3.7. *Let $A \subseteq \mathbb{C}^*$ be a non-empty finite subset and let $W_m := \{z \in \mathbb{C}^* \mid z^m = 1\} = \{e^{\frac{2k\pi i}{m}} \mid 0 \leq k \leq m-1\}$, $m \geq 1$, be the set of m-th roots of unity. We represent A as*

$$A = \bigcup_{i=1}^{s} c_i W_{m_i} \tag{3.1}$$

with $c_i \in \mathbb{C}^*$, $m_i \geq 1$ and such that s is minimal.

(1) *For the group $G := \{\alpha \in \mathbb{C}^* \mid \alpha \cdot A = A\}$ it holds*

$$G = W_d \quad \textit{with} \quad d = \gcd(m_1, \ldots, m_s).$$

In particular, $G \cong \mathbb{Z}_d$ and there exists an $\alpha \in \mathbb{C}^* \backslash \{1\}$ with $\alpha \cdot A = A$ if and only if $d \geq 2$.

(2) *The G-action on A yields a decomposition of A in exactly $m(A) := \frac{m_1 + \cdots + m_s}{d}$ orbits $B_1, \ldots, B_{m(A)}$.*

We will use the following notation: if we write a finite non-empty subset $A \subseteq \mathbb{C}^*$ in the form $A = \bigcup_{i=1}^{s} c_i W_{m_i}$ such that s is minimal (cf. Lemma 3.7), we let

$$d(A) := \gcd(m_1, \ldots, m_s), \quad G(A) := \{\alpha \in \mathbb{C}^* \mid \alpha \cdot A = A\} \cong \mathbb{Z}_{d(A)} \quad \text{and} \quad m(A) := \frac{m_1 + \cdots + m_s}{d(A)}.$$

In addition, if $A = \varnothing$, we let

$$d(A) = d(\varnothing) := 0, \quad G(A) = G(\varnothing) := \mathbb{C}^* \quad \text{and} \quad m(A) = m(\varnothing) := 0.$$

Theorem 3.8 is the main point of this paper:

Theorem 3.8. *Let V be a smooth Gizatullin surface as in Theorem 3.2, (X, D) a 1-standard completion of V and let $A_i = \{P_{i,1}, \ldots, P_{i,r_i}\} \subseteq C_i \backslash (C_{i-1} \cup C_{i+1}) \cong \mathbb{C}^*$, $3 \leq i \leq n-1$, be the base point set of the feathers $F_{i,j}$. Moreover, for $4 \leq i \leq n-2$ let $B_{i,1}, \ldots, B_{i,m(A_i)}$ be the orbits of the $G(A_i)$-action on A_i,*

$$O_{i,j} := \bigcup_{1 \leq l \leq r_i; P_{i,l} \in B_{i,j}} F_{i,l} \cap F_{i,l}^{\vee} \subseteq V, \quad 1 \leq j \leq m(A_i),$$

and

$$O_0 := V \backslash \left(\bigcup_{4 \leq i \leq n-2, 1 \leq j \leq r_i} F_{i,j} \cap F_{i,j}^{\vee} \right).$$

Then the following hold:

(1) *The set O_0 is the big orbit of the action of* $\mathrm{Aut}(V)$ *on V and the subsets $O_{i,j}$ are invariant under* $\mathrm{Aut}(V)$.

(2) *For the fix point set $F(V)$ of the natural action of* $\mathrm{Aut}(V)$ *on V we have*

$$\bigcup_{4 \leq i \leq n-2, d(A_i)=1, 1 \leq j \leq r_i} F_{i,j} \cap F_{i,j}^{\vee} \subseteq F(V).$$

(3) *If at most two of the r_i are non-zero, then O_0 and the $O_{i,j}$ form the orbit decomposition of the natural action of the automorphism group* $\mathrm{Aut}(V)$ *on V. Moreover, equality in (2) holds.*

The most important step for proving Theorem 3.8 is to show that the assertion holds for $\mathrm{Aut}(V, \pi)$ instead of $\mathrm{Aut}(V)$. In other words, we prove the following lemma:

Lemma 3.9. *Let V be a smooth Gizatullin surface as in Theorem 3.2, (X,D) a 1-standard completion of V with induced $\mathbb{A}^1$-fibration $\pi : V \to \mathbb{A}^1$ and let $A_i = \{P_{i,1},\dots,P_{i,r_i}\} \subseteq C_i \backslash (C_{i-1} \cup C_{i+1}) \cong \mathbb{C}^*$, $3 \le i \le n-1$, be the base point sets of the feathers F_{i,j_i}. Moreover, for $4 \le i \le n-2$ let $B_{i,1},\dots,B_{i,m(A_i)}$ be the orbits of the $G(A_i)$-action on A_i,*

$$O_{i,j} := \bigcup_{1 \le l \le r_i; P_{i,l} \in B_{i,j}} F_{i,l} \cap F_{i,l}^{\vee} \subseteq V, \quad 1 \le j \le m(A_i),$$

and

$$O_0 := V \backslash \left(\bigcup_{4 \le i \le n-2, 1 \le j \le r_i} F_{i,j} \cap F_{i,j}^{\vee} \right).$$

Then the following hold:

1. *The set O_0 is the big orbit of the action of* $\operatorname{Aut}(V,\pi)$ *on V and the subsets $O_{i,j}$ are invariant under* $\operatorname{Aut}(V,\pi)$.
2. *For the fix point set $F(V)$ of the natural action of* $\operatorname{Aut}(V,\pi)$ *on V we have*
$$\bigcup_{4 \le i \le n-2, d(A_i)=1, 1 \le j \le r_i} F_{i,j} \cap F_{i,j}^{\vee} \subseteq F(V).$$
3. *If at most two of the r_i are non-zero, then O_0 and the $O_{i,j}$ form the orbit decomposition of the natural action of the automorphism group* $\operatorname{Aut}(V,\pi)$ *on V. Moreover, equality in (2) holds.*

Proof. We denote by $(X, D = C_0 \rhd \cdots \rhd C_n)$ a 1-standard completion of V. We decompose the map $(X,D) \to (\mathbb{F}_1, C_0 \cup C_1)$ into single blowups

$$(X,D) = (X_N, D_N) \overset{\pi_N}{\to} \cdots \overset{\pi_2}{\to} (X_1, D_1) \overset{\pi_1}{\to} (X_0, D_0) = (\mathbb{F}_1, C_0 \cup C_1),$$

such that if two blowups π_s and π_t respectively create feathers $F_{i,\rho}$ and $F_{j,\sigma}$ respectively, then $s < t$ holds if and only if $i < j$ (cf. Remark 2.41).
Let $\psi \in \operatorname{Aut}(V,\pi)$. Then ψ is given on $\mathbb{F}_1 \backslash (C_0 \cup C_1)$ by

$$\psi(x_0,y_0) = (ax_0 + P(y_0), by_0 + c), \quad a,b \in \mathbb{C}^*, c \in \mathbb{C}, P[y_0] \in \mathbb{C}[y_0]$$

Since we can assume that the blowup $(X,D) \to (\mathbb{F}_1, C_0 \cup C_1)$ is centered in $(0,0) \in L_0$, it follows that $P(0) = 0 = c$ and thus

$$\psi(x_0,y_0) = (ax_0 + y_0\tilde{P}(y_0), by_0) \quad \text{for some} \quad \tilde{P}(y_0) \in \mathbb{C}[y_0].$$

We perform coordinate transformations as in Remark 2.41 and claim that the lifting Ψ_i of ψ on the surface (X_i, D_i) has the form

$$\Psi_i(x_i,y_i) = (\alpha x_i(1 + x_i^k y_i^l R(x_i,y_i)), \beta y_i(1 + x_i^k y_i^l S(x_i,y_i))),$$

where $R, S \in \mathbb{C}[[x_i,y_i]]$ are rational functions, expressed as power series and $k \ge 0$, $l,s,t,p,q \ge 1$. Moreover, $k = 0$ holds if and only if we perform no blowups of type 1, that is of type $(x_i,y_i) = (x_{i+1}, x_{i+1}y_{i+1})$ except for the first one in $(0,0) \in L_0$.
We show this by induction. For $i = 0$ the claim is obvious. In the case $(x_i,y_i) = (x_{i+1}y_{i+1}, y_{i+1})$ we obtain after a short computation

$$\begin{aligned}\Psi_{i+1}(x_{i+1},y_{i+1}) &= \left(\frac{\alpha}{\beta} x_{i+1} \cdot \frac{1 + x_{i+1}^k y_{i+1}^{k+l} R(x_{i+1}y_{i+1}, y_{i+1})}{1 + x_{i+1}^k y_{i+1}^{k+l} S(x_{i+1}y_{i+1}, y_{i+1})}, \beta y_{i+1}(1 + x_{i+1}^k y_{i+1}^{k+l} S(x_{i+1}y_{i+1}, y_{i+1}))\right) \\ &= (\tilde{\alpha} x_{i+1}(1 + x_{i+1}^k y_{i+1}^{k+l} \tilde{R}(x_{i+1},y_{i+1}), \beta y_{i+1}(1 + x_{i+1}^k y_{i+1}^{k+l} \tilde{S}(x_{i+1},y_{i+1})))\end{aligned}$$

with $\tilde{\alpha} = \alpha/\beta$ and certain power series $\tilde{R}, \tilde{S}$. Similarly, in the case $(x_i,y_i) = (x_{i+1}, x_{i+1}y_{i+1})$ we obtain

$$\Psi_{i+1}(x_{i+1},y_{i+1}) = (\alpha x_{i+1}(1+x_{i+1}^{k+l}y_{i+1}^{l}\tilde{R}(x_{i+1},y_{i+1}),\tilde{\beta}y_{i+1}(1+x_{i+1}^{k+l}y_{i+1}^{l}\tilde{S}(x_{i+1},y_{i+1})))$$

with $\tilde{\beta}=\beta/\alpha$ and certain power series $\tilde{R},\tilde{S}$. It remains to check the cases, where we create feathers on a certain component C_j. If we create a family of feathers $F_1,\dots,F_m$ on a component C_j with $C_j\backslash C_{j-1}=\{y_i=0\}$ with base points $A=\{\gamma_1,\dots\gamma_m\}\subseteq C_j\backslash(C_{j-1}\cup C_{j+1})\cong\mathbb{C}^*$, we necessary have $\alpha\in G(A)$. Introducing new coordinates (x_{i+1},y_{i+1}) via $(x_i,y_i)=(x_{i+1},y_{i+1}\cdot Q(x_{i+1}))$ with $Q(x_{i+1})=\prod_1^m(x_{i+1}-\gamma_j)$, we obtain

$$\begin{aligned}\Psi_{i+1}(x_{i+1},y_{i+1}) &= (\alpha x_{i+1}\cdot(1+x_{i+1}^{k}Q(x_{i+1})^{l}y_{i+1}^{l}R(x_{i+1},Q(x_{i+1})y_{i+1})),\\ &\quad \frac{\beta y_{i+1}Q(x_{i+1})\cdot[1+x_{i+1}^{k}Q(x_{i+1})^{l}y_{i+1}^{l}S(x_{i+1},Q(x_{i+1})y_{i+1})]}{Q(\alpha x_{i+1}\cdot(1+x_{i+1}^{k}Q(x_{i+1})^{l}y_{i+1}^{l}R(x_{i+1},Q(x_{i+1})y_{i+1})))})\\ &= (\alpha x_{i+1}\cdot(1+x_{i+1}^{k}Q(x_{i+1})^{l}y_{i+1}^{l}R(x_{i+1},Q(x_{i+1})y_{i+1})),\\ &\quad \frac{\beta}{\alpha^m}\cdot\frac{y_{i+1}Q(x_{i+1})\cdot[1+x_{i+1}^{k}Q(x_{i+1})^{l}y_{i+1}^{l}S(x_{i+1},Q(x_{i+1})y_{i+1})]}{Q(x_{i+1}\cdot(1+x_{i+1}^{k}Q(x_{i+1})^{l}y_{i+1}^{l}R(x_{i+1},Q(x_{i+1})y_{i+1})))})\\ &= (\alpha x_{i+1}(1+x_{i+1}^{k}y_{i+1}^{l}\tilde{R}(x_{i+1},y_{i+1}),\tilde{\beta}y_{i+1}(1+x_{i+1}^{k}y_{i+1}^{l}\tilde{S}(x_{i+1},y_{i+1})))\end{aligned}$$

with $\tilde{\beta}=\frac{\beta}{\alpha^m}$ and certain power series $\tilde{R},\tilde{S}$. Similarly for the case where we create feathers with base points on the y_i-axe.

We fix an arbitrary index $3\le j\le n-1$. To create the feathers on the component C_j we blow up r_j points on $C_j\backslash(C_{j-1}\cup C_{j+1})$. By an appropriate choice of coordinates we can assume that the base points A_j lie on the x_N-axis, i.e. $P_{j,i}=(\gamma_i,0)$, $1\le i\le r_j$. Since ψ lifts to X we have $\Psi(A_j)=A_j$. Ψ induces on $C_j\backslash(C_{j-1}\cup C_{j+1})$ the multiplication $x\mapsto\alpha\cdot x$, i.e. $\alpha\in G(A_j)$. We consider points $P_{j,s}$ and $P_{j,t}$ such that $\Psi(P_{j,s})=P_{j,t}$, i.e. such that $\alpha\cdot\gamma_s=\gamma_t$ holds. After the blowup in $P_{j,s}$ and $P_{j,t}$ respectively we introduce affine coordinates (u,v) and (u',v') respectively via $(x_N,y_N)=(uv+\gamma_s,v)$ and $(x_N,y_N)=(u'v'+\gamma_t,v')$ respectively. We represent the arguments of Ψ_N in the coordinates (u,v) and the images in the coordinates (u',v'). In these coordinates we have $F_s\backslash D=\{v=0\}$ and $F_t\backslash D=\{v'=0\}$. By a short computation we see that the lifting $\tilde{\Psi}$ of Ψ_N, after a blowup on the left-hand side in P_s and on the right-hand side in P_t, has the form

$$\tilde{\Psi}(u,v) = \left(\frac{\alpha}{\beta}u\cdot(1+v^l\tilde{R}(u,v))+v^{l-1}\tilde{S}(u,v),\beta v\cdot(1+v^l\tilde{T}(u,v))\right)$$

for certain power series $\tilde{R},\tilde{S},\tilde{T}$. Thus we see that $\tilde{\Psi}|_{F_s\backslash D}:F_s\backslash D\to F_t\backslash D$ is given by

$$\tilde{\Psi}(u,0)=\begin{cases}\left(\frac{\alpha}{\beta}u+\tilde{S}(u,0),0\right) & ,l=1\\ \left(\frac{\alpha}{\beta}u,0\right) & ,l\ge 2.\end{cases}$$

We claim that for $l\ge 2$ the points $(u,v)=(0,0)$ and $(u',v')=(0,0)$ respectively are precisely the intersection points $F_s\cap F_s^\vee$ and $F_t\cap F_t^\vee$ respectively, if $F_s^\vee$ and $F_t^\vee$ are matching feathers for F_s and F_t after a reversion centered in $p=(\lambda:1:0)\in C_0\backslash C_1$. We consider once again the coordinates (x,y) and (x_i,y_i) on 0-*standard pairs* to describe the affine pieces U_i defined in Section 2.5. A simple computation shows that (x,y) coincides with our initial coordinates (x_0,y_0) if and only if $\lambda=0$. Thus if $p=(0:1:0)$, the points $(u,v)=(0,0)$ and $(u',v')=(0,0)$ respectively are the points $F_s\cap F_s^\vee$ and $F_t\cap F_t^\vee$ respectively (cf. Lemma 2.40 and Remark 2.41). But since $\mathrm{Aut}(X,D)$ acts transitively on $C_0\backslash C_1$ and every $\psi\in\mathrm{Aut}(X,D)$ fixes these points for $l\ge 2$, we can assume that the reversion is centered in $p=(0:1:0)$. Now we can use the description given in Section 2.5 and the claim follows by Remark 2.41. Thus we have $\Psi(F_s\cap F_s^\vee)=F_t\cap F_t^\vee$ for all $\psi\in\mathrm{Aut}(V,\pi)$ if and only if $l\ge 2$. Therefore, for $l\ge 2$ the sets $O_{j,k}$ are invariant under the action of $\mathrm{Aut}(V,\pi)$.

Now we show that $l\ge 2$ holds if and only if $j\le n-2$. We consider the construction of the extended divisor D_{ext} starting from the divisor $C_0\rhd C_1\rhd L$ on $\mathbb{F}_1$. We denote the exponents k,l

in the i-th lifting Ψ_i now by k_i, l_i. The first blowup in $(0,0) \in L_0$ is of type 1 and we obtain $k_1 = 0$ and $l_1 = 1$. During further blowups we can first perform as many blowups as possible of type 2. During these steps we obtain $k_i = 0$, $l_i = 1$ (see induction step!). Here the right-hand boundary component gives the y_i-axis and the next-to-last one gives the x_i-axis. If the feathers $F_{j,k}$ are attached to C_{n-1}, we need no more blowups and we end up in $l = l_N = 1$. But if $3 \le j \le n-2$ holds, we need at least one more transformation of type 1 and thereafter at least one transformation of type 2 (since the base points of the feathers $F_{j,k}$ lie on the x_N-axis). Transformations of type 1 give $k_{i+1} = k_i + l_i$ and $l_{i+1} = l_i$ and transformations of type 2 give $k_{i+1} = k_i$ and $l_{i+1} = k_i + l_i$. It follows that after performing all these transformations we have $k_i \ge 1$ and $l_i \ge 2$. In particular we obtain $l_N \ge l_i \ge 2$.
Assertion (2) is an immediate consequence of (1) since $d(A_i) = 1$ implies that the sets $O_{i,j}$ are points.
The proof of (3) is the same as the one of Theorem 3.2 (3). $\square$

Remark 3.10. We consider a smooth Gizatullin surface V as in Theorem 3.2, a 1-standard completion (X, D) of V and a reversion $\varphi : (X, D) \dashrightarrow (X^\vee, D^\vee)$ centered in $p \in C_0 \backslash C_1$. If $F_{i,j}$ and $F^\vee_{i,j}$ form a pair of matching feathers, the map φ leaves the points $p_{i,j} := F_{i,j} \cap F^\vee_{i,j}$ invariant. Indeed, φ induces the identity on the affine part $V = X\backslash D \cong X^\vee \backslash D^\vee$ and identifies the feathers $F_{i,j}$ with their proper transforms (this can also be seen by considering the minimal resolution $(X, D) \leftarrow (Z, B) \rightarrow (X^\vee, D^\vee)$ of φ). However, we have to pay attention that the feathers $F^\vee_{i,j}$ themselfs (as algebraic curves) depend on p in general. In the special case of Theorem 3.2, the subgroup $\mathrm{Aut}(X, D)$ leaves the intersection points $p_{i,j}$ invariant, but *not* the curves $F^\vee_{i,j}$ themself.

We now show Theorem 3.8:

Proof of Theorem 3.8: First we notice that $F_{3,i}\backslash D$ and $F_{n-1,i}\backslash D$ is contained in the big orbit of $\mathrm{Aut}(V)$. Indeed, if a feather F is attached to C_{n-1} then automorphisms $\psi(x_0, y_0) = (x_0 + ay_0, y_0)$, $a \in \mathbb{C}$, induce translations on F (this corresponds to the case $l = 1$ in the proof of Lemma 3.9). The same argument yields that there are automorphisms in $\mathrm{Aut}(V, \pi^\vee)$ that induce translations every feather attached to C_3.
Let now $\varphi \in \mathrm{Aut}(V)$. We extend φ to a birational map $\varphi : (X, D) \dashrightarrow (X, D)$, which belongs either to $\mathrm{Aut}(X, D)$ or admits a decomposition $\varphi = \varphi_m \circ \cdots \circ \varphi_1$, where each φ_i is either a reversion or a fibered modification. Thus φ decomposes into a sequence

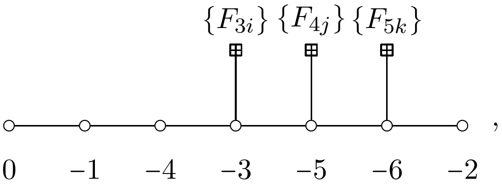
$$(X, D) = (X_0, D_0) \overset{\varphi_1}{\dashrightarrow} (X_1, D_1) \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_m}{\dashrightarrow} (X_m, D_m) = (X, D).$$

Since any 1-standard completion of V is isomorphic either to (X, D) or to $(X^\vee, D^\vee)$, we can assume that every (X_i, D_i) is either (X, D) or $(X^\vee, D^\vee)$.
If $\varphi_i : (X, D) \dashrightarrow (X, D)$ is a fibered modification, then by Lemma 3.9, the subsets $O_{i,j}$ are invariant under φ_i. The same holds for fibered modifications $\varphi_i : (X^\vee, D^\vee) \dashrightarrow (X^\vee, D^\vee)$. We notice that by the Matching Principle the orbit decomposition of the action of $\mathrm{Aut}(V, \pi)$ is the same as the one of the action of $\mathrm{Aut}(V, \pi^\vee)$, if the feathers are not contained in the big orbit O of $\mathrm{Aut}(V)$.
If $\varphi_i : (X, D) \dashrightarrow (X^\vee, D^\vee)$ (or $\varphi_i : (X^\vee, D^\vee) \dashrightarrow (X, D)$) is a reversion, then the subsets $O_{i,j}$ are also invariant under φ_i (cf. Remark 3.10). Therefore, the map φ leaves the subsets $O_{i,j}$ invariant. This ends the proof.

$\square$

Remark 3.11. If V admits more than two families of feathers, then the orbit decomposition of $\mathrm{Aut}(V)$ becomes in general more complicated and there may be a strict inclusion in Theorem 3.8 (2). The problem is obvious: we have only two parameters, which induce motions $x \mapsto \alpha_i \cdot x$ on the $*$-components C_i, $4 \le i \le n-2$ (these parameters are given by $a, b \in \mathbb{C}^*$ in the fibered modifications $\psi(x_0, y_0) = (ax_0 + y_0 Q(y_0), by_0)$). Moreover, it is difficult to control these motions

since they cannot be read out directly from the form of the extended divisor.
We consider the following example: let V be as in Theorem 3.2 with extended divisor



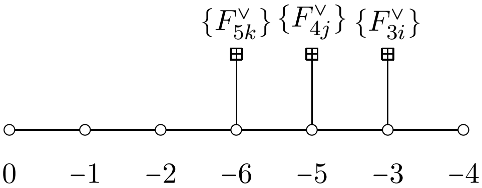
$1 \le i \le 2$, $1 \le j \le 3$, $1 \le k \le 4$ and the base points of the feathers should be organized as follows:

$$A_3 := W_2 = \{\pm 1\}, A_4 := W_3 = \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}, A_5 := W_4 = \{\pm 1, \pm i\}.$$

We consider the surface (X', D'), which is obtained by blowing down all the feathers of (X, D). The same computations as in the proof of Lemma 3.9 show that the Jonquieres automorphisms $\psi(x_0, y_0) = (ax_0 + y_0 Q(y_0), by_0)$ induce the following motions on the boundary components C_i':

(1) $x \mapsto a^4 b^{-1} \cdot x$ on $C_3' \backslash (C_2' \cup C_4')$,
(2) $x \mapsto a^3 b^{-1} \cdot x$ on $C_4' \backslash (C_3' \cup C_5')$,
(3) $x \mapsto a^2 b^{-1} \cdot x$ on $C_5' \backslash (C_4' \cup C_6')$.

Here we identify $C_i' \backslash (C_{i-1}' \cup C_{i+1}')$ with $\mathbb{C}^*$ in a way that $\{\infty\} = C_i' \cap C_{i-1}'$ and $\{0\} = C_i' \cap C_{i+1}'$ holds. ψ can be lifted to an automorphism of V if and only if the sets A_i are invariant under ψ. A simple computation shows that A_3 and A_5 are invariant under ψ if and only if $a = b = 1$ or $a = b = -1$. In both cases, ψ induces the identity on C_4' and therefore it fixes all points in A_4. It follows that $\mathrm{Aut}(V, \pi)$ acts trivial on O_4.
Now we consider the motions on the boundary components after reversion. After performing a reversion (with an arbitrary center) the extended divisor of V has the form



with $1 \le i \le 2$, $1 \le j \le 3$, $1 \le k \le 4$. By the Matching Principle the base point sets are of the form

$$A_3^\vee = c_1 W_4, A_4^\vee = c_2 W_3, A_5^\vee = c_3 W_2 \quad \text{for appropriate} \quad c_1, c_2, c_3 \in \mathbb{C}^*.$$

Let $((X^\vee)', (D^\vee)')$ be again the surface obtained by blowing down all the feathers of $(X^\vee, D^\vee)$. A short computation yields that the Jonquieres automorphisms $\psi(x_0, y_0) = (ax_0 + y_0 Q(y_0), by_0)$ induce the following motions on the boundary components:

(1) $x \mapsto a^2 b^{-1} \cdot x$ on $(C^\vee)_3' \backslash ((C^\vee)_2' \cup (C^\vee)_4')$,
(2) $x \mapsto a^3 b^{-2} \cdot x$ on $(C^\vee)_4' \backslash ((C^\vee)_3' \cup (C^\vee)_5')$,
(3) $x \mapsto a^4 b^{-3} \cdot x$ on $(C^\vee)_5' \backslash ((C^\vee)_4' \cup (C^\vee)_6')$.

Again we see that the conditions $\psi(A_3^\vee) = A_3^\vee$ and $\psi(A_5^\vee) = A_5^\vee$ imply that $a^3 b^{-2} = 1$. Thus ψ induces the identity on $(C^\vee)_4' \backslash ((C^\vee)_3' \cup (C^\vee)_5')$ and all points in $A_4^\vee$ are fixed under the action of $\mathrm{Aut}(V, \pi^\vee)$.
Thus the orbit decomposition under the action of $\mathrm{Aut}(V, \pi)$ as well as of $\mathrm{Aut}(V, \pi^\vee)$ is given by

$$O_0 \;=\; V \backslash \left(\bigcup_j F_{4,j} \cap F_{4,j}^\vee \right), O_1 = \{F_{4,1} \cap F_{4,1}^\vee\}, O_2 = \{F_{4,2} \cap F_{4,2}^\vee\}, O_3 = \{F_{4,3} \cap F_{4,3}^\vee\}.$$

The same arguments as in the proof of Theorem 3.8 show that this is already the orbit decomposition under the action of $\mathrm{Aut}(V)$. In particular, the points $F_{4,j} \cap F_{4j}^\vee$, $1 \le j \le 3$, are fix points of the action of $\mathrm{Aut}(V)$, although we have $d(A_4) = 3$.

Conversely, we need D_{ext} to admit $*$-components to exhibit surfaces with a non-transitive action of the automorphism group. Indeed, the following proposition holds:

Proposition 3.12. *Let V be a smooth Gizatullin surface, $(X, D = C_0 \triangleright \cdots \triangleright C_n)$ a 1-standard completion of V, such that the components $C_2, \ldots, C_n$ are $+$-components. Then* $\mathrm{Aut}(V)$ *acts transitively on V.*

Proof. We will only sketch the proof, leaving the details to the reader. We consider a 1-standard completion of V such that every feather is attached to its mother component (by Corollary 6.1.3 in [FKZ6] such completions always exist). Fixing a feather F with mother component C_t, we choose an appropriate 1-standard completion (X, D) of V, where F is attached to the last component C_n (we can force F to jump by an appropriate choice of reversions and fibered modifications of the reversed completion, cf. [FKZ6], Section 4.3). Now a simple computation shows that the automorphisms

$$\varphi(x_0, y_0) = (x_0 + a y_0^{t-1}, y_0), \quad a \in \mathbb{C},$$

of $\mathbb{A}^2 = \mathbb{F}_1 \backslash (C_0 \cup C_1)$ lift to automorphisms of (X, D) and induce non-trivial translations on $F \backslash D \cong \mathbb{A}^1$, depending on the continuous parameter a. Thus all points of $F \backslash D$ have an infinite orbit and are therefore contained in the big orbit of the action of $\mathrm{Aut}(V)$. This implies that $\mathrm{Aut}(V)$ acts transitively on V. □

In particular, since Danilov-Gizatullin surfaces V_{k+1} can be completed by a zigzag of type $[[0, -1, (-2)_k]]$ containing only $+$-components, their automorphism group acts transitively.

Remark 3.13. Let V be a normal affine variety. We denote by $\mathrm{SAut}(V)$ the subgroup of $\mathrm{Aut}(V)$ generated by all algebraic subgroups isomorphic to the additive group $\mathbb{G}_a$. Then a point $x \in V$ is called *flexible*, if the tangent space $T_x V$ is spanned by the tangent vectors of the orbits $H.x$ of one-parameter unipotent subgroups $H \subseteq \mathrm{Aut}(V)$. Moreover, we call V flexible if every point $x \in V_{\text{reg}}$ is flexible.
If V is a flexible affine variety then $\mathrm{SAut}(V)$ acts transitively on V_{reg}. Moreover, in [AFKKZ] it was shown that for a normal affine variety V of dimension ≥ 2 the following conditions are equivalent (cf. [AFKKZ], Theorem 0.1):

(1) X is flexible.
(2) $\mathrm{SAut}(V)$ acts transitively on V_{reg}.
(3) $\mathrm{SAut}(V)$ acts infinitely transitively on V_{reg}.

The last condition means that for any collection of points $\{P_1, \ldots, P_k\}$ and $\{Q_1, \ldots, Q_k\}$ in V_{reg} there exists an automorphism $\varphi \in \mathrm{SAut}(V)$ such that $\varphi(P_i) = Q_i$.
Therefore, instead of "flexible" we sometimes say "infinitely transitive". Similarly, V is called *stably infinitely transitive* if $V \times \mathbb{A}^m$ is infinitely transitive for some $m \geq 0$.
Theorem 3.8 shows that smooth Gizatullin surfaces V with a distinguished and rigid extended divisor are *not flexible* in general. But the following question arises:

Question/Problem: Let V be a smooth Gizatullin surface with a distinguished and rigid extended divisor. Is V stably infinitely transitive?

3.3. **The amalgamated product structure of the automorphism group.** The automorphism groups of surfaces considered in Theorem 3.2 can be represented as amalgamated products of certain subgroups.

Corollary 3.14. *Let V be as in Theorem 3.2. We choose a fixed $\mathbb{A}^1$-fibration $\pi : V \to \mathbb{A}^1$ and consider the corresponding $\mathbb{A}^1$-fibration $\pi^\vee : V \to \mathbb{A}^1$, induced by the reversion $\psi : (X, D) \dashrightarrow (X^\vee, D^\vee)$ with center $p \in C_0 \backslash C_1$.*

(1) *If $\mathcal{F}_V$ consists of a vertex and a loop, i. e. $(X, D) \cong (X^\vee, D^\vee)$, then the automorphism group of V is the free product of $A = \langle \mathrm{Aut}(X, D), \psi \rangle$ and $J = \mathrm{Aut}(V, \pi)$, amalgamated over their intersection $A \cap J = \mathrm{Aut}(X, D)$:*

$$\mathrm{Aut}(V) = A \star_{A\cap J} J.$$

(2) *Let $\mathcal{F}_V$ be of the form $\mathcal{F}_V :\ [(X,D)] \bullet\!\longleftrightarrow\!\bullet\ [(X^\vee, D^\vee)]$. We denote by A the subgroup corresponding to the edge and by J and $J^\vee$ the subgroups $J := \mathrm{Aut}(V,\pi)$ and $J^\vee := \mathrm{Aut}(V,\pi^\vee)$. Identifying $J \supseteq A \subseteq J^\vee$ we have*

$$\mathrm{Aut}(V) = J \star_A J^\vee.$$

Proof. Let $\mathcal{F}_V$ have a loop. First we show that $\mathrm{Aut}(V)$ is generated by $\mathrm{Aut}(X,D), \psi$ and J. We can extend every automorphism $\varphi \in \mathrm{Aut}(V)$ to a birational map $\varphi : (X,D) \dashrightarrow (X,D)$. Then φ either belongs to $\mathrm{Aut}(X,D)$ or it can be decomposed as

$$\varphi = \varphi_n \circ \cdots \circ \varphi_1 : (X,D) = (X_0,D_0) \overset{\varphi_1}{\dashrightarrow} (X_1,D_1) \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_m}{\dashrightarrow} (X_m,D_m) = (X,D),$$

where every φ_i is a fibered modification or a reversion. Since every 1-standard completion of V is isomorphic to (X,D), we can assume that $(X_i,D_i) = (X,D)$. Therefore, every element φ_i can be considered as an element of $\mathrm{Aut}(V)$. If φ_i is a fibered modification, then we have $\varphi_i \in J$. But if g_i is a reversion, then it can be written as $\varphi_i = \alpha_i\psi\beta_i$ with certain $\alpha_i, \beta_i \in \mathrm{Aut}(X,D)$, since $\mathrm{Aut}(X,D)$ acts transitively on $C_0\backslash C_1$. Thus we obtain $\mathrm{Aut}(V) = \langle \mathrm{Aut}(X,D), \psi, J\rangle = \langle A, J\rangle$.
Now we write any $\varphi \in \mathrm{Aut}(V)$ as $\varphi = a_n \circ j_n \circ \cdots \circ a_1 \circ j_1$ with $a_i \in A\backslash J$ and $j_i \in J\backslash A$. Then a_i is a product of reversions which is not an isomorphism, and j_i is a fibered modification. Theorem 2.16 then yields that $\varphi \notin \mathrm{Aut}(X,D)$. Thus it follows that $\mathrm{Aut}(V) = A \star_{A\cap J} J$.
Assertion (2) follows immediately from Remark 2.23. □

4. The singular case

Our results can be generalized to the case of singular Gizatullin surfaces. First, we have to generalize the notion of a $*$-component.

Definition 4.1. (1) *For a general feather F with dual graph*

$$\Gamma_F :\quad \begin{array}{ccccccc} \circ & \!\!\text{——}\!\! & \circ & \!\!\text{——}\!\! & \cdots & \!\!\text{——}\!\! & \circ \\ B & & D_1 & & & & D_k \end{array}$$

and bridge curve B we call D_k the tip component of F

(2) *The component C_i is called a* $*$-component *if*
 (i) *$D_{ext}^{\geq i+1}$ is not contractible and*
 (ii) *$D_{ext}^{\geq i+1} - F_{j,k}$ is not contractible for every feather $F_{j,k}$ of $D_{ext}^{\geq i+1}$ such that the tip component of $F_{j,k}$ has mother component C_τ with $\tau < i$.*

Otherwise C_i is called a $+$-component.

It is easy to see that also in the non-smooth case, that is where feathers can have more than one component, $*$-components appear as a result of an inner blowup of the previous zigzag, while an outer blowup of a zigzag creates a $+$-component. Similarly as in the smooth case we have the following lemma:

Lemma 4.2. *Let D_{ext} be the extended divisor of the minimal resolution of singularities of a 1-standard completion of a certain Gizatullin surface V. Suppose that every C_i, $3 \leq i \leq n-1$, is a $*$-component and that there are no feathers attached to the component C_n. Then every feather $F_{i,j}$ is an A_k-feather, that is every $F_{i,j}$ is contractible and therefore has the dual graph*

$$\Gamma_{F_{i,j}} :\quad \begin{array}{ccccccc} -1 & & -2 & & & & -2 \\ \circ & \!\!\text{——}\!\! & \circ & \!\!\text{——}\!\! & \cdots & \!\!\text{——}\!\! & \circ \\ B & & D_1 & & & & D_k \end{array},$$

with k depending on i and j.

Note that especially for A_k-feathers the mother components of all curves $D_1, \dots, D_k$ coincide since any A_k-feather is born by successive blowups of a point on the boundary component it is attached to.

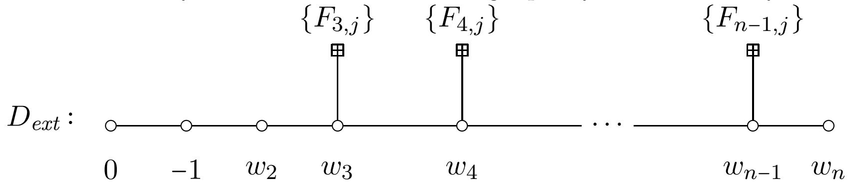
Remark 4.3. If every component C_i, $3 \le i \le n-1$, of D_{ext} is a $*$-component and if there are no feathers attached to C_2 and C_n, then it is easy to see that the same property holds for the extended divisor $D^{\vee}_{\mathrm{ext}}$ after reversion (with an arbitrary center).

In the following we will always assume that the following condition holds:

($*$) V admits a $1-$standard completion (X, D) such that $C_3, \dots, C_{n-1}$ are $*$-components and there are no feathers attached to C_2 and C_n.

We can generalize Theorem 3.2 as follows:

Theorem 4.4. *Let V be a Gizatullin surface as in ($*$) with $n \ge 4$ and let (X, D) be the minimal resolution of singularities of a 1-standard completion of V . For every $s \ge 0$, we let $A_{i,s} = \{P_{i,s,1}, \dots, P_{i,s,r_{i,s}}\} \subseteq C_i \backslash (C_{i-1} \cup C_{i+1}) \cong \mathbb{C}^*$, $3 \le i \le n-1$, be the base point set of the feathers $F_{i,j}$, $1 \le j \le r_i$, which are A_s-feathers. Then the dual graph of D_{ext} has the form*



with $1 \le j \le r_i$. Moreover, the following hold:

(1) *For any two 1-standard completions (X', D') and (X'', D'') of V with $(X' \backslash D', \pi') \cong (X'' \backslash D'', \pi'')$ we have $(X', D', \bar{\pi}') \cong (X'', D'', \bar{\pi}'')$.*

(2) *The graph $\mathcal{F}_V$ has one of the following two forms:*

$$\mathcal{F}_V : [(X, D)] \bullet \longleftrightarrow \bullet [(X^{\vee}, D^{\vee})] \quad \text{or} \quad \mathcal{F}_V : [(X, D)] \bullet \circlearrowleft .$$

If $\mathcal{F}_V$ is of the form $\bullet \circlearrowleft$, then $D^{\ge 2}$ is a palindrome and there exist elements $\gamma_i \in \mathbb{C}^*$, $3 \le i \le n-1$, such that

$$A_{i^{\vee},s} = \gamma_i \cdot A_{i,s} \quad \text{holds for all } 3 \le i \le n-1, s \ge 0.$$

(3) *If $r_i > 0$ holds for at most two indices $i \in \{3, \dots, n-1\}$, then $\mathcal{F}_V$ has the form $\bullet \circlearrowleft$ if and only if $D^{\ge 2}$ is a palindrome and there exist elements $\gamma_i \in \mathbb{C}^*$ such that*

$$A_{i^{\vee},s} = \gamma_i \cdot A_{i,s} \quad \text{holds for all } 3 \le i \le n-1, s \ge 0.$$

(4) $\mathrm{Aut}(V)$ *is generated by automorphisms of $\mathbb{A}^1$-fibrations if and only if $\mathcal{F}_V$ has no loops except for the case $\Gamma_D = [[0, -1, -2, -2, -2]]$. If $\Gamma_D = [[0, -1, -2, -2, -2]]$, then V is smooth.*

Proof. Let (X', D') and (X'', D'') be two 1-standard completions of V such that $(X' \backslash D', \pi') \cong (X'' \backslash D'', \pi'')$. By Lemma 2.12 such an isomorphism is given by a Jonquieres automorphism of the form $\psi(x_0, y_0) = (ax_0 + y_0 P(y_0), by_0)$, $P(y_0) \in \mathbb{C}[y_0]$ of $\mathbb{A}^2 = \mathbb{F}_1 \backslash (C_0 \cup C_1)$. The same computation as in the proof of Theorem 3.2 shows that ψ lifts to an automorphism of a 1-standard completion (X, D) of V as well as to the minimal resolution of singularities (Y, B) of (X, D) (notice that $B \cong D$ since D is contained in the regular locus of X). Now, a similar computation as in the proof of Theorem 3.2 shows that ψ describes on $D^{\ge 2}$ (before creating the feathers by blowups of points on the boundary components) the same map as the isomorphism $\tilde{\psi}(x_0, y_0) = (ax_0 + cy_0, by_0)$ with $c = P(0)$. By Lemma 3.1 we obtain that $(X', D') \cong (X'', D'')$. This shows (1).

The proof of assertion (2), (3) and (4) is the same as in the smooth case. $\square$

Remark 4.5. (The generalized Matching Principle, cf. [FKZ6], Remark 3.3.9) We can generalize the Matching Principle for feathers of arbitrary length instead of length one. The generalized matching Principle provides a one-to-one correspondence between feathers $F_{i\rho}$ and $F^{\vee}_{j\sigma}$ such that the mother component of the bridge curve $B_{i\rho}$ of $F_{i\rho}$ is equal to the mother component of the tip of $F^{\vee}_{j\sigma}$.
In particular, if V is a Gizatullin surface satisfying $(*)$, then every feather $F_{i\rho}$ and $F^{\vee}_{j\sigma}$ is an A_k-feather and thus all components of $F_{i\rho}$ (and $F^{\vee}_{j\sigma}$) have the same mother component (it is just the boundary component the feather is attached to).

Now we can generalize Theorem 3.8 to the singular case.

Theorem 4.6. *Let V be a Gizatullin surface as in Theorem 4.4, (X,D) a 1-standard completion of V and let $\mu:(Y,D)\to(X,D)$ be the minimal resolution of singularities of (X,D). For every $s\geq 0$, we let $A_{i,s}=\{P_{i,s,1},\dots,P_{i,s,r_{i,s}}\}\subseteq C_i\backslash(C_{i-1}\cup C_{i+1})\cong\mathbb{C}^*$, $3\leq i\leq n-1$, be the base point set of the feathers $F_{i,j}$, $1\leq j\leq r_i$, which are A_s-feathers, and $A_i:=\bigcup_{s\geq 0}A_{i,s}$. Moreover, we let $F_{i,s,1},\dots,F_{i,s,r_{i,s}}$ denote those feathers which are A_s-feathers and which are attached to C_i.*

(1) *Let $G_i:=\bigcap_{s\geq 0}G(A_{i,s})$. Moreover, for $4\leq i\leq n-2$ let $B_{i,s,1},\dots,B_{i,s,m_{i,s}}$ be the orbits of the G_i-action on $A_{i,s}$,*

$$O_{i,s,j}:=\bigcup_{1\leq l\leq r_{i,s};P_{i,s,l}\in B_{i,s,j}}\mu(F_{i,s,l})\cap\mu(F^{\vee}_{i,s,l})\subseteq V,\quad 1\leq j\leq m_{i,s}, s\geq 0,$$

and

$$O_0:=V\backslash\left(\bigcup_{4\leq i\leq n-2,j}\mu(F_{i,j})\cap\mu(F^{\vee}_{i,j})\right).$$

Then the set O_0 is the big orbit of the action of $\mathrm{Aut}(V)$ *on V and the subsets $O_{i,s,j}$ are invariant under* $\mathrm{Aut}(V)$.

(2) *For the fix point set $F(V)$ of the natural action of* $\mathrm{Aut}(V)$ *on V we have*

$$\bigcup_{4\leq i\leq n-2,\ j,\ \gcd(d(A_{i,s})|s\in\mathbb{N})=1}\mu(F_{i,j})\cap\mu(F^{\vee}_{i,j})\subseteq F(V).$$

Here $\gcd(d(A_{i,s})\mid s\in\mathbb{N})$ *denotes the well-defined natural number*

$$\gcd(d(A_{i,0}),d(A_{i,1}),d(A_{i,2}),\dots,d(A_{i,k})),$$

where $d(A_{i,s})=0$ for all $s>k$.

(3) *If at most two of the r_i are non-zero, then O_0 and the $O_{i,s,j}$ form the orbit decomposition of the natural action of the automorphism group* $\mathrm{Aut}(V)$ *on V. Moreover, equality in (2) holds.*

Proof. The proof is precisely the same as in the smooth case. □

Remark 4.7. Corollary 3.14 holds as well for arbitrary Gizatullin surfaces satisfying $(*)$, since the proof uses only the structure of the graph $\mathcal{F}_V$.

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